AN INTRODUCTION TO THE GEOMETRY OF THE FOURFOLD

कालखात्मदिशां सर्व्वगतत्वं परमं महत् ।

—भाषापरिच्छेदः ।

BY

SURENDRA MOHAN GANGULI, D.Sc.
LECTURER IN HIGHER GEOMETRY AND STATISTICS
IN THE UNIVERSITY OF CALCUTTA

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To

SYAMAPRASAD MOOKERJEE, Esq.

M.A., B.L., Bar.-at-Law, M.L.C.

In Love and Admiration

PREFACE

The present work embodies the course in the geometry of four dimensions which, for several years, it has been part of my duties as Lecturer in Mathematics at the University of Calcutta to give to the Post-Graduate students of the Pure Mathematics Department. In the preparation of the work, therefore, my attempt has chiefly been confined to a legitimate extension of two or three dimensional geometries, with introduction of new notions for proper explanation of higher dimensional configurations. The nature of discussion has, accordingly, been twofold: firstly, those properties in lower spaces have been selected, which lend themselves easily to generalisation in higher spaces; and secondly, those have been discussed which are peculiar to higher spaces and logically present themselves as necessary interpretations of algebraic equations in more than three variables, having no corresponding analogues in lower spaces.

Introduction of analytical ideas in the field of geometry has greatly widened the scope of geometrical researches in higher spaces, and, as observed by M. Poincaré, the objects of hyperspaces are susceptible to precise definitions as those of the ordinary space. Although they cannot be represented, we can conceive them and study their properties. Following a logical and systematic course of analytical investigation, we become familiar with various strange phenomena, not to be perceived but to be conceived and reconciled

by free play of imagination in forming their mental pictures. An attempt has, therefore, been made to harmonise geometry with analysis in the method of treatment. With a view to avoid tedious and lengthy processes, it has been found convenient to maintain generally an analytical method, with occasional use of intuitive geometrical concepts. What is principally aimed at is to acquaint the student with new facts, rather than exercise him in the application of principles already acquired in studying lower geometries.

Hypergeometry is too wide in its scope, admitting of various developments in diverse directions. It is, therefore, absurd to expect, within the limited compass of a work of an introductory character, to adequately deal with the entire subject, or to do full justice to the different topics selected. No attempt has, consequently, been made to produce a complete systematic treatise on the subject, but certain representative topics in the fourfold have been selected, sufficient to indicate in which directions generalisation of theorems in lower geometries is possible, and at the same time, to reveal strange and unexpected phenomena, having no analogues in lower spaces. Topics chosen are expected to give a general outline of the growth of knowledge of hypergeometry so as to enthuse the beginner with a keen desire for peeping into the mysteries of a four-dimensional world and of hyperspaces generally.

The notion of a higher dimensional space is difficult to conceive at an early stage, but when once introduced as a necessity for suitable interpretation of algebraic equations in more than three variables, the mind becomes gradually trained and accustomed to grasp the notion,

Thus, discussion of topographical properties and interpretation of equations form the main characteristics of the earlier chapters. Four dimensional concepts have been gradually introduced in later stages, when sufficient familiarity has already been acquired with the nature of the hyperspace. In order to familiarise the student with the nature of hypersurfaces, surfaces and curves in the fourfold, some of their general properties have been discussed, with brief statements specially of their curvature properties, in the last three chapters, mainly by the application of differential methods.

The literature on the subject of hypergeometry has enormously increased in recent years. A fairly complete history of the development of the subject from the earliest times is to be found in Professor Manning's Geometry of Four Dimensions, and other references up to 1911 can be had from Professor Sommerville's Bibliography. Rudiments of the subject, however, have been given by P. H. Schoute in his two volumes of *Mehrdimensionale Geometrie*. There are so many workers in the field and such a large number of papers are annually published in different languages that it will be presumptuous to claim any of these results as my own, but some of them were published in 1914 and 1922 in two small book forms, and the rest, inspired by the classical works of Veronese, Bertini, Jordan, Jouffret, Segre, Schläfli, Schoute, Cayley and others, and the celebrated papers of Kommerell and Cole, were obtained during these years of lecture work. In the preparation of the present work, constant recourse has been had to the works of these authors, and free use has been made of their results with necessary references in proper places. My

obligations to these mathematicians, perhaps greater than I can confess, are gratefully acknowledged. During the final revision of the manuscript, Professor Forsyth's two volumes of *Geometry of Four Dimensions* (published in 1930) were consulted with much advantage, and in some cases, specially in the differential portion, methods of proof were somewhat modified on his lines. Exact terminology in the geometry of higher spaces having not yet been settled, many of the technical terms, very ingeniously coined by him, besides those adopted by Schoute and others, are adopted in the present work. Properties of isocline planes have been discusssd on the lines of Manning and Stringham. Chapter IX is chiefly, if not entirely, based on Kommerell's well-known paper. A number of exercises have been inserted to illustrate the properties under discussion, and references have been traced to standard works and to original sources as far as practicable.

In concluding this preface, I must once more record my deep debt of gratitude to the illustrious Asutosh Mookerjee, whose spirit has always been a living inspiration to me in continuing the work undertaken during his lifetime. It is, indeed, gratifying to note that the work should be completed at a time when his worthy son, Mr. Syamaprasad Mookerjee, M.A., B.L., Bar.-at-Law, M.L.C., our youngest Vice-Chancellor and President of the Post-Graduate Councils, occupies the position of his father in the University of Calcutta.

Finally, I must thank my friend and colleague, Prof. S. N. Mitter, M.A., for his assistance in the preparation of the work and the authorities of the University who have kindly undertaken its publication. I must

also thank the Superintendent, Mr. A. C. Ghatak, M.A., and the staff of the Calcutta University Press for the extreme care and promptness with which the printing was finished within a very short time.

For several reasons the work had to be hurried through the Press. From the nature of the work, it is not unlikely that, besides typographical errors, other errors of omission and commission have crept into its pages. Any correction or suggestion for the improvement of the work will be very much appreciated.

UNIVERSITY OF CALCUTTA:
September, 1934.

S. M. G.

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CHAPTER I

PRELIMINARY NOTIONS

1. Notion of Dimensions:

The universe we inhabit and those we surmise to be near it are all contained in what is called " Space " or *Âkásha*, according to the Hindus. This is supposed to contain all we perceive infinite before us and in whatever directions we cast our eyes. Knowledge of material objects is intuitive and develops by a co-ordination of our senses of sight and touch. The crude notions of shape, bulk, etc., came to be analysed gradually and three " dimensions " were attributed to all material objects and the infinite space containing them, without knowing fully why or how this peculiar number " three " was selected. This is perhaps the result of collective experience for a long time which supplies this hereditary idea, and it may be called the " axiom " of the dimensions. The reason for this particular selection may be explained in the following manner * :—

The conception of space is based on the image formed on the retina of our eyes, which is said to be

* H. Poincaré, l'Espace et la Géométrie, Rev. de métaph. et de morale, 1895 and 1897.

of two dimensions. The notion of a third dimension arises from the effort at accommodation, which is made by our eyes, and the convergence given to them. It depends upon these two distinct indications which are really concordant and equivalent to only *one*. These two elements of our visual sensations co-operate in forming the notion of space and are then certainly functions of *three* independent variables, and it is from this that the usual conception of three dimensions arise.

Thus, the ideas of shape and size of material objects with superficial extent and length gradually developed and were made abstract; and then necessity arose for their proper representation, which led to the conception of geometrical figures. As the power of abstraction grew finer, conceptions were made of surfaces apart from the material objects, and this gave rise to the conception of a region of two dimensions within the space, which is conceived to be of three dimensions. The idea of dimensionality was thus formed when Greek Geometry started, and this ultimately led to a finer conception of a region of one dimension, namely, length. Thus, a line was said to be of one dimension, a plane or surface of two dimensions and the space of three dimensions.

2. Notion of Higher Dimensions:

Human mind took a long time to adjust itself to the new conception of the three dimensions—length, breadth and thickness—by means of which the contents of a cubic body, and in fact of all material bodies, were

determined and expressed in mathematical terms. After these conceptions were fully realised and the mathematical study of space, plane and line was far advanced, there arose a tendency to conceive of a higher dimension beyond the three, but still the conception was, and is even now, confined to the comparatively very small class of mathematicians and philosophers.

The logic underlying this conception of a fourth dimension may be explained by the fact that the two indications constituting the notion of three dimensions, as stated above, may be regarded quite independent of each other, without being connected by any relation whatsoever, and a fourth dimension can be attributed to our visual space. This conception has again been used by some investigators to explain certain superphysical phenomena, which seem otherwise inexplicable.

3. Analytical Conception of Space:

From an analyst's viewpoint, the space is nothing but a system of three variables called "distances" or "co-ordinates," to which are assigned all imaginable independent values, and the phenomena of nature are interpreted accordingly. A point of space is consequently regarded as a system of values determined from three variables, and the knowledge of the material world as that of the relations between them and with 'time' under certain circumstances called 'phenomena.' The works of Laplace, Lagrange, Poisson, and others are all based on this theory. The ultimate effects of these three variables may then be regarded as the causes which act around us and are

at the root of all visible phenomena. Its real significance, confusedly perceived by the ancients, must therefore be settled in the long run under a form, which is more or less independent of what is called 'space,' endowed with three dimensions, and is a concrete substitute for the abstract analytic system. The analyst, therefore, did not remain content with only three variables. His method being more and more forceful and exacting, he has carried his researches further into the world of small dimensions, which was ignored by the ancients, but discovered by a deeper study from results, the causes of which seemed non-existent till then. Accordingly, for purposes of further studies, the system of three variables has been replaced by one of four. The fourth variable, however, appears of very small magnitude and of very slight motion, and consequently, becomes concrete by a very small fourth dimension ; and for a minute study of nature, the number of variables requires to be increased and consequently that of the spatial dimensions. This viewpoint, if considered reasonable, will then produce in our mind a transformation corresponding to that of analysis, attended with a sensation of visual conception of space with four dimensions.

4. The Hindu Theory of Successive Spaces :

The Hindu Philosophy is more emphatic on the existence of multiple worlds constituting an expanse called *Saṁsára*,* which consists of various grades of experiencing beings inhabiting the various universes.

* Shârîraka Vâshya, II. i. 36.

The existence of the *sensible* in regard to the universe is admitted no doubt, but then the originating sources of the sensible are themselves *supersensible* realities, constituting other worlds, which may be called 'transcendental.' Hence, there are as many varieties of worlds as there are fundamentally different types of beings, forming themselves into a number of grades and orders ; these latter again form a series, in which *Man* comes as one extremity with most limited experience. It is a matter of common knowledge that man is often frustrated or unexpectedly helped in his endeavours mysteriously by unseen powers, and it must, therefore, be admitted that there are beings, higher and more powerful than man, who exist in unseen supersensible forms in worlds which are supersensible. Thus, in the *Saṁsára* there are different worlds and these beginningless and endless series of universes are all linked together by an inexorable law of cause and sequence, and under this law the ingredients are eternally alternating between the phases of *chaos* and *cosmos*. The two periods, constituting what is called a '*kalpa*,' succeed one another by virtue of *Kála* (time), and as *Kála* urges the created things, it is *Dik* (direction), acting in opposition to *Kála*, which holds them in their relative positions. According to Hindu realism, both *Kála* and *Dik* exist as *realities*. *Kála* creates and moves things off and gives rise to the notion of past, present and future, of old and new, and produces all temporal relations and is, in this sense, called 'Time.' On the other hand, *Dik*, acting in opposition to *Kála*, maintains relative positions of things at all moments of time, and gives rise to the notion of far and near, in this direction

or that direction, etc., and is, in this sense, 'relation of special direction.' Thus, it appears that the Hindu realism recognises *Kála* and *Dik* as realities, which hold together the sensible universe in the infinite space of *Ákásha*, which ever moves on in well-regulated and seasoned cycles, and yet maintains that positional order which obtains among its members. Western Philosophy, however, declares that space and time are pure abstractions. They are neither real entities nor attributes thereof, but merely orders of co-existing and successive phenomena. This exposition of time and space, however, in no way carries us far into the mathematical investigation with which we shall be engaging ourselves presently.

5. Mathematical Conception of Time and Space:

The first mathematical conception of time and space is to be traced to a lecture by Barrow, Lucasian Professor of Mathematics, in which he attempted a rational discussion of time from the mathematical standpoint. According to him, there is great affinity and analogy between space and time. As space is to magnitude, so does time seem to be to motion ; so time is some sort of space of motion. The analogy of time, space and motion also places time in the category of measurable quantities, the measure being made by motion ; Barrow, however, does not support identification of time and a line, as they are quantities of different kinds, but he has no objection to the analogy, if that is not extended to identity, which is not based on logic. The notion of time plays an important rôle in Mathematical Physics and a quantitative measure of time is often essential

in this branch of knowledge. Although time should not be identified with a line, its representation by a line is perfectly justified. Since time consists of parts altogether familiar, it is logical to regard it as a quantity endowed with one dimension only ; for it may be imagined to be made up either of the simple addition of 'rising moments' or of the 'continual flux' of one moment, and in consequence, only length is ascribed to it, and its quantity may be determined by the length of the line passed over. Just as a line is looked upon as the trace of a moving point, and its quantity consists of but one length following the motion, time may also be conceived as the trace of a moment continually flowing and its quantity pursues but one succession, stretched out, as it were, in length, determined and exhibited by the length of the space moved over.

This representation of time by a line is a first step towards a mathematical theory and is very useful in ordinary Physics, but becomes inadequate in modern Physical Sciences, where the conception of atomism framed in space-time co-ordinates is used. Some of the recent researches in Mathematical Physics seem to show that time actually consists, in so far as physical phenomena are concerned, of 'atoms of time,' having values of the order of $4{\cdot}5 \times 10^{-24}$ seconds, *i.e.*, four and a half million million millionths of a second. Contemporaries and followers of Newton, and other mathematicians of the eighteenth century gave more or less philosophical discourses on time and space.* Works of Euler, Maclaurin, and others, referring to time and space, all lead

* Isis, Vol. 19, where a good account is given of the history of mathematical time.

to the view which is essentially held by modern men of science, namely, "there is something beyond all our science and mathematics." These serve only to indicate the nature of the physical world, and we must turn to other methods, if we would discover the nature of its system. Whatever might be the nature of time and space and their relations, it can be easily maintained that in graphical representation of the natural phenomena, time plays a very important rôle and may accordingly be regarded as a co-ordinate similar to the three *spatial* co-ordinates—an accepted principle in the theory of Relativity.

6. Space-time Co-ordinates in the Theory of Relativity :

The dependence of time-readings on the co-ordinate system, to which it is referred, has led to the conclusion that time must be introduced as a fourth co-ordinate along with the three spatial co-ordinates. In graphical representation, time is always regarded as a co-ordinate. Since all spatial observations occur at some time or other, and all time-occurrences are bound up with spatial phenomena, it may be conceived that all events transpire for us in a four-dimensional world, in which three dimensions belong to space and the fourth to time. The indivisible linking up of space and time in all physical phenomena, however, has been recognised for the first time as an absolute necessity by the realisation of the relativity of time. A space-time representation * in the theory of relativity

* H. Minkowski's lecture, "Raum und Zeit." Abdruck in Lorentz-Einstein-Minkowski, "Das Relativitätsprinzip." Leipzig (1915).

has accordingly been developed so far as to represent all events in space and time by a four-dimensional geometry in a space-time continuum. The four-dimensional continuum is called the 'world' and x, y, z, t, the co-ordinates of a "world-point;" the continuum of x, y, z alone is referred to as "space." In order to give time the outward appearance of a fourth world co-ordinate, it is conceived as the path traversed by light in time t, thus allowing time to appear explicitly to interpret the results easily. The law of constancy of light-velocity in vacuo is stated in the form

$$x_1^2+x_2^2+x_3^2-x_4^2=0$$

where x_1, x_2, x_3 are the co-ordinatesof a world-point, represented on a cartesian co-ordinate system, and $x_4=ct$, where c represents the velocity of light in vacuo. How the time co-ordinate x_4 is to be related to the three space co-ordinates x_1, x_2, x_3 is essential in discerning the relation between the four-dimensional space-time geometry and four-dimensional Euclidean geometry. We note that the quadratic expression $x_1^2+x_2^2+x_3^2-x_4^2$ is unaltered by linear homogeneous transformation in four-dimensional space-time continuum, and represents the square of the distance of the world-point from the origin. If, then, the time co-ordinate $x_4=ct$ is regarded as a real quantity, laid off on an imaginary axis in a fourth dimension at right angles to the three space co-ordinates, *i.e.*, if we put $x_4=ict$ (and $c=1$), the above expression reduces to $x_1^2+x_2^2+x_3^2+x_4^2$, which represents the square of the distance of the point (x_1, x_2, x_3, x_4) from the origin in the four-dimensional Euclidean geometry. Thus, we find that the four-dimensional space-time geometry is closely related to Euclidean

geometry, but not simply a four-dimensional Euclidean geometry. The choice of an imaginary time-co-ordinate is very useful, as it has the advantage of generalising formulae of three-dimensional geometry and directly extending the results to the space-time geometry in connection with the theory of relativity. Non-Euclidean geometry, however, must be developed on the original assumption $x_4 = ct$.

7. Geometrical Conception of Spaces:

The geometrician conceives of space as divided into an infinite number of 'slices' of infinitely small thickness, which are called 'planes;' these planes are again supposed to be divided into an infinity of 'strips,' extremely thin, which are called 'lines;' and these lines again are conceived as divided into an infinity of 'segments,' infinitely small, and are called 'points.' These planes, lines and points are not exactly slices, strips and segments, but they really mean *separations* apart from any notion of space, thickness and reality. These divisions, which separate one from the other, are considered as positive matters with or without the things contained therein. Sections are freely made and they are filled with all sorts of imaginable figures and laws are applied to them. The geometrician says that among these dimensions, there is *one* near the plane, there are *two* near the line and *three* near the point; or what is the same thing, planes have *two* dimensions, the line has only *one*, and the point has *none*. In fact, these partitions are purely ideal and can be effected in an infinite number of ways.

This conception can, in an analogous manner, be extended logically on the supposition that our world is only one of the infinitely *thinned* slices, forming so many parallel slices in an expanse of four dimensions, and in such an expanse these constituent spaces can be filled, displaced and intersected in all sorts of manner. Such an expanse, formed by this infinite number of thinned slices and in which these are situate, is the " Fourfold," just as each of these spaces contains an infinity of planes, each of these latter contains an infinity of lines, and each of these latter again an infinity of points. The analogy may be further extended and the process is perfectly logical from the geometrician's viewpoint.

8. Perspective View of the Spaces :

The partitions conceived above are mere abstractions. Consider the horizontal plane we walk on, supposed infinite in all directions. The things that may be conceived as lying in this are nothing but the sections made by it across the bodies of three dimensions in the space containing it. Thus, the lines and points constituting figures in the plane exist only in imagination through an abstraction of the third dimension. These figures are conceived as freely displaced in the plane, but in the sense of a third dimension, they have a thickness, which is nil or infinitely small, and they have no real existence, as bodies of three dimensions have. In an analogous manner, the space of our experience is conceived to be in an expanse of four dimensions, compared to which it is like a slice, infinitely thinned and totally flat, just like a plane in

the space. The fourth dimension is nil or infinitely small. Solid material objects (figures) will be the intersections of the space with the bodies of four dimensions, not existing wholly within it and having real existence in the expanse of four dimensions. These four-dimensional bodies then will have real existence, and all others, namely, solids, surfaces, lines and points will be mere abstractions without any real existence. The solids then will have a thickness infinitely small or nil in the sense of the fourth dimension. They will then appear to the eye as the space on the surface, just as the plane is found beneath our feet. The space of our experience will then not be an absolute and unique thing, but a simple unit among an infinite number. We conceive all these, but we are not in a position to give any reason for or against the existence of spaces congeneric to ours, or the existence of the fourfold expanse containing them all. We cannot visualise the entire fourfold, and consequently, it is without interest for all but the geometrician or the Metaphysicist. These *thinned* spaces together with ours form a single and identical continuum, just as the planes of our space do. This view of the space is very useful in explaining many intricate matters in the physical sciences.

In this sense and this sense only, the geometrician considers the world to belong to a greater expanse called the *Fourfold*, if this latter exists at all; and he applies all the geometrical laws here, just as there are geometrical laws in our space and the plane.

9. Mathematical Treatment:

The mode of treatment of four-dimensional entities is one of projection and section. Just as figures in

space may be represented by means of projections on a plane or planes, the configurations in the fourfold can be represented by projections on our space. Relief models of surfaces and polyhedroids are constructed and often used, but these constructions are complicated and require to be supplemented by descriptive geometry. However much a solid form can be conceived through its projections on planes, it is absolutely impossible to form a concrete notion of the forms in a fourfold from their projections on our space, without conceiving the same by other means. This is just like a blind man perceiving the different parts of the body of an elephant, but at the same time not being able to form a definite idea of the body as a whole. Similarly, none of the projections and sections of these four-dimensional bodies will help us in any way to realise them in concrete forms.

Although we are not in a position to visualise these forms in the fourfold, we are not thereby precluded from studying their geometry, *i.e.*, descriptive and metric relations existing between them. Many strange and mysterious phenomena will logically appear to exist in the fourfold, which may, in consequence, be called the *fairyland* of mathematics or the *Spirit-world*. But at the same time, it must not be assumed that the geometry of higher dimensions has no intrinsic interest apart from being a good ground for mathematical exercises. It exerts a strong influence on modern sciences where it has a recognised position. It has widened the ordinary geometrical ideas, disclosed new aspects of theories relating to planes and space and supplied new methods of investigation, not known before, yielding many new results and has finally simplified

many difficulties in the theory of numbers, functions, substitutions, etc., with simple and expressive interpretation of abstract matters.

All these advantages, derived from a logical extension of our geometrical conceptions to higher spaces, point to only one conclusion, namely, that if really there is a fourth dimension, we are blind to it, but at the same time, we should not be bound by these limitations. We are perfectly justified in extending our bounds of knowledge by a series of logical reasoning and peeping into the mysteries of hyper-spaces with ease, if they are consistent with, and not contradictory to, our experience. Mathematicians often admit and speak of infinitely great numbers, imaginary roots of equations, etc., and make use of these to the best advantage, but they never admit concrete reality for them, nor discuss it even. Similarly, a geometrician is quite free to formulate laws in his own way, provided they are consistent and logical and never contradict the accepted principles of logic. With these reservations, we shall often speak of the extent and the objects of the fourfold, as if these really existed; and with a view to facilitate exposition, we shall freely use expressions, figures, constructions, etc., implying real existence.

10. Application in Modern Physics:

The province of physical sciences is made up of concrete objects of nature, whereas we have so far discussed only objects of reasoning without any reference to their objective existence. If, however, a four-dimensional space be conceived and the objects thereof be assumed to have real existence, the ordinary space with all its objects of one, two or three dimensions are

things having real existence ; and for this purpose, one additional dimension, but almost negligible compared with those already existing, is to be assumed. Portions of space, plane, line and even the point will now be portions of the fourfold, having respectively one, two, three and four additional dimensions, all very small ; and in this way, we shall have nothing but abstract objects. With these conventions, rules of geometry of four dimensions can now be applied to these objects, which will supply very simple and strictly rational explanations of various phenomena, the direct causes of which cannot be ascertained by these sciences.

Ancient physics introduced vague, unthinkable entities, which were neither less transcendental nor less intangible than the fourth dimension. We had to accept hundreds of arbitrary, artificial, complicated and even contradictory hypotheses, taxing too much our power of conception—such as the 'formless' matter of Aristotle, the 'subtle matter' of Descartes, the 'Monads' of Leibnitz, 'elastic and solid ether' of Fresnel, the 'electrons' of Larmor, etc. The axioms and principles of ordinary space were not much helpful in explaining the nature of the hypothetical things, such as atoms, atomic vibrations, molecules and molecular movements, etc. The real nature of these things remained a mystery to us. The modern Physical Sciences, on the other hand, try to explain these by considering them as movements of material atoms. The problems become much simplified, if, in addition to the three components of movements and forces already applied to these atoms, a fourth is applied in a direction at right angles to all of them. This component forms a quadri-rectangular system with those three. Thus, the physicist finds it extremely easy

to speak of the multiple vibrations manifested under the names 'heat,' 'light,' 'electricity,' etc. This extension of the science of Mechanics simplifies matters much more strikingly than the older hypothesis. The fact that the fourth component appears only in the ultra-microscopic field suggests that the form of our universe should possess a certain extremely small extension in the sense of the fourth dimension.

11. Representation of Hyper-spaces:

The notion of higher dimensional spaces dates back from the time of Aristotle and arose out of necessity for geometrical representation of mathematical concepts. Apart from other considerations, Gauss measure of curvature of a curved surface necessitates the existence of an uncurved space, more extensive in dimensions than the curved configuration. If, therefore, the ordinary space of three dimensions is assumed to possess curvature, the mathematical conception of such curvature would require the existence of some other space of dimensions higher than three, characterised by the property of being completely uncurved. The existence of such an objective space must be established by observation and through measurement. Mathematical physicists have often asserted the objective existence of such a space, time being regarded as a fourth dimension. Einstein's Theory of Relativity, as already stated, has given an impetus to the notion of time being regarded as co-ordinate in quality with the conceptions of the ordinary space of three dimensions. However much the mathematician finds it easy to postulate an additional dimension of his abstract space, he never deals

with topics other than the properties of such a space, nor does he ever assert the objective existence thereof. His sole object is to interpret mathematical formulae representing relations in nature. It must be clearly understood that for graphical representation, time as a variable may be represented along a line, simultaneously with the three variables denoting distances—length, breadth and thickness,—but the term 'dimension' cannot logically be used to denote the conception of range of space as well as that of range of time ; for, these four constituent variables do not constitute an irresoluble order of four co-ordinate dimensions, the last being clearly of a different nature from the former three. It follows hence that time may be very convenient for graphic illustration in mathematical investigations, but it never constitutes a dimension in the sense of the three dimensions of ordinary space.

Mathematical expressions for the aggregate of geometrical entities demand four independent variables and in consequence their configuration is frequently termed four-dimensional. The word 'dimension,' as used in such investigations, is a mere substitute for the word 'variable' only for graphic representation, there being not the slightest suggestion for the objective existence of a four-dimensional space. The mathematician is, therefore, concerned only with the abstract geometry of any number of dimensions without discrimination among them as regards significance or capacity for representing suitable magnitudes and without any reference to the objective space. We, therefore, assume that the dimensions are co-ordinate among themselves and a logical and consistent system of geometry may be developed with the help of certain fundamental notions and relations suitable for these spaces.

12. Nature of Abstract Geometry :

Abstract geometry of any number of dimensions is based on a number of fundamental notions and postulates, both explicit and implicit. In offering explanations of the physical universe, new definitions are assumed and calculations based on these definitions and postulates lead to new results. The explanatory theory is then regarded as a working hypothesis, but it is to be noted that these are all mere conventions, conveniently assumed for purposes of calculation, and can never be regarded as embodying established truths.

The science of abstract geometry is treated accordingly in the two following ways :—

(1) as a natural extension of the ordinary two- and three-dimensional geometries ;

(2) as a need in these higher geometries and in analysis generally, with formulation of notions which have no analogues in the lower geometries.

When two or three variable quantities are connected together in any manner, the nature of the relation or relations is frequently made more intelligible by regarding them as co-ordinates of a point in plane or in space. If there are more than three variables so related, the case becomes more complex and necessity arises for their proper representation by means of a space of proper dimensionality with a complete system of geometry of its own. In the following pages we shall try to develop the properties of the *Fourfold*, without any reference to its objective existence.

CHAPTER II

INTERPRETATION OF EQUATIONS

13. Characteristic Properties of the Fourfold :

The abstract geometry of the fourfold is based upon certain fundamental notions derived in the gradual development of the geometries of two and three dimensions. The primary fundamental notion is that of a point, which is irresoluble and void of all properties except position. The fundamental relation assumed existing between points is that of collinearity, *i.e.*, lying on a straight line, which is a secondary notion. The complete range of infinitude of points connected by this relation of collinearity is said to constitute a 'linear aggregate,' and the linear aggregate is said to be constituted of all its points.* A point in the fourfold has four degrees of freedom, and can, therefore, belong to two, three or four distinct linear aggregates. The ensemble of points is said to form a *Continuum* or *Manifold*, which may be of one, two, three or four dimensions. A linear aggregate is, therefore, a manifold of one dimension, which has only one degree of freedom and is called a 'straight line,' limited or unlimited in extent. Two distinct and independent linear aggregates are said to constitute a manifold of

* Thus, a linear aggregate is completely determined when we take any two points, all points collinear with them and all points collinear with any two obtained in this way.

two dimensions, having two degrees of freedom, which is a *plane* in the ordinary Euclidean geometry. Three distinct and independent linear aggregates are said to constitute a manifold of three dimensions, having three degrees of freedom, which may be called a *Hyperplane*. Finally, four distinct linear aggregates are said to constitute the manifold of four dimensions, *i.e.*, the *Fourfold*, having four degrees of freedom. Thus, the dimension of any manifold is specified by the number of independent linear aggregates which compose it, or into which it may be resolved and the manifolds are said to be *linear*. The fourfold, then, contains linear manifolds of one, two or three dimensions, which are respectively termed *lines*, *planes* and *hyperplanes*, limited or unlimited in extent.

From what has been stated above, it is clear that the fourfold is characterised by the two following fundamental characteristic properties :—

(1) The straight line joining any two points of any manifold lies entirely in that manifold ;

(2) Through any point can be drawn *four* and only *four* independent lines, no three of which lie in one plane. It follows hence that through any point can be drawn four and only four mutually perpendicular lines forming an orthogonal system.

14. Loci in the Fourfold :

Taking an unrestricted point as the primary fundamental entity, it has degrees of freedom equal in number to the dimensions of the fourfold, namely, four. One degree of restraint imposed removes one degree

of freedom, and every additional degree of restraint removes another degree of freedom. Thus, in the Euclidean space of three dimensions, a restraint in the form of an equation picks out only those points which have two degrees of freedom, and consequently lie in a region called a *surface*, or in special circumstances, a *plane*. An additional restraint imposed by means of another equation will remove another degree of freedom, and only the points, having only one degree of freedom, will thereby be picked out ; and these will lie on a *curve*, or in particular cases, on a *straight line*. Finally, three equations will destroy all freedom and will give one or more discrete points. Exactly in a similar manner, in the fourfold, one restriction imposed by means of an equation will destroy one degree of freedom and the points picked out will lie in a three-dimensional space, which we shall call a *hyper-surface*, or in particular circumstances, a *hyper-plane*. Two equations will remove two degrees of freedom and will, therefore, denote points having two degrees of freedom, namely, those lying on a *surface* or a *plane* ; three equations will determine the points having only one degree of freedom, namely, those lying on a *curve* or a *straight line*. Four equations will completely destroy all the four degrees of freedom and will determine only one or more discrete points.

From what has been postulated in the preceding article, it is now easy to reconcile the two methods of specifying the same manifold. Thus, a point having four degrees of freedom may be specified by means of four independent variables, to which all imaginable values can be given, or which may be

connected by four independent equations. A linear aggregate, having only one degree of freedom, has lost its other three degrees of freedom, and this loss may be specified by means of three equations; or, in othe r words, a line may be specified by means of three equations between four variables. Again, two linear aggregates constituting a surface (or plane) have two degrees of freedom only and lost two other degrees of freedom, the loss being specified by two equations; or, in other words, a surface (or plane) may be defined by means of two equations between four variables. Finally, three linear aggregates constituting a hyper-surface (or hyperplane) have three degrees of freedom and lost only one degree of freedom, the loss being specified by only one equation; or, in other words, a hyper-surface (or hyperplane) may be defined by means of only one equation between four variables.

15. Algebraic Discussion:

A point in the fourfold is, therefore, specified by means of four independent variables, x, y, z, w. These may be regarded as a set of characteristic parameters, generally used in much of modern investigations in abstract geometry, or they may be regarded as the cartesian co-ordinates, referred to the system of four mutually orthogonal lines that may be drawn through a fixed point as origin.

In view of the considerable difficulty that presents itself in conceiving the existence of two or more hyperplanes in the fourfold, it is inconvenient to consider the reference-system for the present. We shall, however, content ourselves with the discussion of these

matters from the viewpoint of Algebra, and see what possible geometrical interpretations can be offered for algebraic equations involving four variables.

Suppose the four co-ordinates of a point are connected by the following four simultaneous linear equations :—

$$\left.\begin{aligned} A &\equiv a_1x+b_1y+c_1z+d_1w+e_1=0 \\ B &\equiv a_2x+b_2y+c_2z+d_2w+e_2=0 \\ C &\equiv a_3x+b_3y+c_3z+d_3w+e_3=0 \\ D &\equiv a_4x+b_4y+c_4z+d_4w+e_4=0 \end{aligned}\right\} \qquad (15.1$$

These equations are assumed all distinct and mutually independent, *i. e.*, the matrix of their co-efficients is not zero. They can, therefore, be solved for the four variables and will give unique values for them. Consequently, these four linear equations can be regarded as specifying the position of a point in the fourfold corresponding to the common set of values of the variables. The above four equations completely destroy all freedom of the point, which becomes thereby fixed.

It may be pointed out that if one or more of the above four equations are of higher orders, they will specify more than one discrete points.

Next, consider the three equations :—

$$A=0, \quad B=0, \quad C=0. \qquad (15.2$$

These three equations destroy three degrees of freedom of the point (x, y, z, w), and consequently, one degree of freedom is left, and the point generates a linear aggregate, which is of one dimension

only. In fact, the three equations cannot give unique values of the four variables but will determine their ratios, and consequently they are now functions of a single variable parameter. The points thus specified, therefore, compose what has been previously described as a *linear aggregate*, or a straight line.

Hence, three linear equations in four variables can geometrically be interpreted as representing a linear aggregate.

It should be noted that when one or more of these equations are of higher orders, they specify a one-dimensional locus (a curve) in the fourfold.

Next, consider two of these equations, namely,

$$A=0, \qquad B=0. \tag{15.3}$$

Here, the equations destroy two degrees of freedom and only two degrees of freedom are left. Consequently, the variables can be expressed as functions of two independent parameters, and therefore admit of doubly infinite sets of values specifying two linear aggregates, which constitute a linear manifold of two dimensions, *i.e.*, a plane ; or, what is the same thing, two linear equations can be graphically represented by means of a plane.

Here again, one or both may be of higher orders and in that case, they will specify a manifold of two dimensions—a surface (and not a plane) of two dimensions.

Finally, consider only the equation

$$A=0. \tag{15.4}$$

Here, the equation puts only one restraint upon the point and one degree of freedom is thereby lost. The point has then three degrees of freedom, and the variables can be expressed as functions of three independent parameters and therefore admit of triply infinite systems of values specifying three independent linear aggregates. These, then, constitute a manifold of three dimensions, *i.e.*, a hyperplane ; or, in other words, a linear equation in four variables is geometrically represented by a 'hyperplane.'

If, however, the equation is of higher order, it denotes a manifold of three dimensions, *i.e.*, a hypersurface (and not a hyperplane) of three dimensions.

If the variables (x, y, z, w) are subject to no condition, they are free to admit quadruply infinite systems of values, and the locus is the linear manifold of four dimensions, *i.e.*, the fourfold.

16. A New Aspect of the Configurations:

The above exposition clearly indicates that a linear aggregate or a line contains a single infinitude of points, the points being supposed separated from each other by a finite (but infinitely small) interval.

The surface (or plane) may be considered as being generated by the motion of a line, moving parallel to itself, having one of its points on another line. It consists, therefore, of a doubly infinite system of points. The hyper-surface (or hyperplane) can similarly be considered as generated by the motion of a surface (or plane) moving parallel to itself, having one of its points in a fixed line, which has no other point common with it. The hyper-surface (hyperplane),

therefore, consists of a triply infinite system of points.

Finally, the fourfold may be regarded as being generated by the motion of a hyperplane, moving parallel to itself, having one of its points in a fixed line, which has no other point common with it. The fourfold, therefore, consists of a quadruply infinite systems of points.

From these considerations, a point as an element of a line may be said to have an infinitely small length, dx. The line as an element of a plane may be said to have a second infinitely small second dimension, dy. The plane as an element of the hyperplane may be said to have a third infinitely small dimension, dz. Finally, the hyperplane as an element of the fourfold may be said to have a fourth infinitely small dimension, dw. This aspect then clearly agrees with what has been explained in § 8.

17. Geometric and Analytic Point:

As explained before, a point may be defined by means of four equations between four variables, and these may be regarded as a more general system of co-ordinates referred to the four corresponding hyperplanes which meet at the point. This exposition is analogous to the representation, in the geometries of two or three dimensions, by means of co-ordinate systems and their transformation. The methods employed and the calculations are all independent of the orientation of the system of axes. This presentation is common to all cartesian geometries and affords a convenient mode of study for geometries of higher

spaces, and is based on the equations, which are called the equations of the point. The point is an element of all geometrical configurations, just like an atom of all material bodies. There is, however, an *analytic entity*, constituted by the values of the four variables, which is capable of corresponding with anything we please. These entities, all independent, form a continuum such that, each one of them always corresponds to a given set of values of the four variables, and conversely, each set of values corresponds to an entity, and the ensemble of all these will be analytically equivalent to the fourfold. The values are regarded as the co-ordinates of each individual entity, which is called a 'point.' These individuals are all mutually independent, and their number is the same as that of the points in the fourfold.

This analytic point, however, is not the same as that of the ordinary geometry, which is a *geometrical entity*, clearly defined as the limit of a volume, its dimensions being regarded infinitely small. It is regarded as a portion of a very small cubic meter, as has already been suggested (§ 7), and is always regarded as such.

18. Two Linear Equations:

Consider the two linear equations:—

$$A=0, \qquad B=0. \tag{15.3}$$

Each of these equations represents a hyperplane, and, therefore, $A+\lambda B=0$, where λ is a parameter, represents a hyperplane for all values of λ. Since the values of the variables, which satisfy both the

equations $A=0$ and $B=0$, also satisfy the equation $A+\lambda B=0$, whatever be the values of the parameter λ, we conclude that the hyperplane $A+\lambda B=0$ contains the common points of the two hyperplanes $A=0$ and $B=0$, or, in other words, $A+\lambda B=0$ represents a singly infinite system of hyperplanes through the common points of $A=0$ and $B=0$, which, as we have already seen, determine a plane. Thus, we find that the two hyperlanes $A=0$ and $B=0$ have common between them a plane which is common to all hyperplanes of the system $A+\lambda B=0$, and this latter may be said to represent the "generating hyperplanes" of the plane.

Hence, *any two hyperplanes intersect in a plane and through their intersection an infinity of hyperplanes can be passed.*

19. Three Linear Equations:

Consider the case of three hyperplanes :—

$$A=0, \quad B=0, \quad C=0.$$

We have seen that the values of the variables, which simultaneously satisfy these three linear equations, determine a singly infinite set of points lying on a straight line. Hence, we may conclude that the three hyperplanes intersect in one and the same line.

The equation $A+\lambda B+\mu C=0$, where λ and μ are parameters, represents a doubly infinite system of hyperplanes through the common points of the three constituent hyperplanes, namely, the straight line, and consequently, it denotes the 'generating net of hyperplanes' of the line.

In fact, the three equations $A=0$, $B=0$, $C=0$, taken two by two, represent three planes, all of which again pass through a common line.

The same three equations admit of another interpretation, namely, each pair determines a plane and the remaining represents a hyperplane, which intersects this plane in a line.

Thus, we conclude that *any three hyperplanes intersect in a line, and a plane and a hyperplane also intersect in a line.*

20. Four Linear Equations :

We have already seen that four linear equations specify a point uniquely. Hence, the four hyperplanes represented by these four equations have only one point common, *i.e., four hyperplanes intersect in only one point.* But four such simultaneous equations admit of other interpretations as well.

(1) Consider the line represented by the three equations $A=0$, $B=0$ and $C=0$, and the hyperplane represented by the equation $D=0$.

The common values of the variables satisfying these four equations determine a point uniquely. Therefore, the line determined by the first three equations has only one point common with the hyperplane represented by the last equation, *i.e.,* the line intersects the hyperplane in only one point. Thus, we conclude that *a line and a hyperplane in the fourfold intersect in only one point.*

Note that the line cannot meet the hyperplane in any other point ; for, in that case, the line will lie entirely in the hyperplane.

(2) Consider the two planes represented by the two pairs of equations:

$$\left.\begin{array}{l} A=0 \\ B=0 \end{array}\right\} \quad (20.1) \qquad \left.\begin{array}{l} C=0 \\ D=0 \end{array}\right\} \quad (20.2)$$

If the above two planes have any common point or points, their co-ordinates must satisfy the four linear equations simultaneously. But the above four equations give only one unique set of values of the variables. Hence, we conclude that the two planes represented by the above equations can have only one common point.

Thus, *any two planes in the fourfold intersect in only one point*. It is to be noted that this is a strange phenomenon evidenced only in the fourfold, and is contrary to our experience in the ordinary space of three dimensions, where any two planes always intersect in a line.

If, however, the four linear relations reduce to three independent ones, no unique solution can be obtained, and three of the variables can be expressed in terms of the *fourth*, to which again an infinity of values can be given. In this case, the two planes have a common line of section, and consequently, they lie in one and the same hyperplane. The generating hyperplanes of the two planes will in this case be identical, and consequently the constants will be proportional.

Thus, $A+\lambda B=0$ and $C+\mu D=0$ will be identical and the conditions for this are—

$$\frac{a_1+\lambda b_1}{c_1+\mu d_1}=\frac{a_2+\lambda b_2}{c_2+\mu d_2}=\frac{a_3+\lambda b_3}{c_3+\mu d_3}$$

$$=\frac{a_4+\lambda b_4}{c_4+\mu d_4}=\frac{a_5+\lambda b_5}{c_5+\mu d_5}=\nu,$$

whence eliminating λ, μ, ν, we obtain the necessary conditions in the form—

$$\left\| \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \end{matrix} \right\| = 0$$

21. Five or more Linear Equations :

(1) Consider the case of a plane represented by the equations—

$$A=0 \text{ and } B=0,$$

and a line represented by the three equations—

$$C=0, \quad D=0, \quad E=0.$$

There are then five linear equations involving four variables, and therefore these equations cannot, in general, be satisfied by a common set of values of the variables, unless the co-efficients are connected by some relation. In fact, any four of the five equations will determine a unique set of values of the four variables, which, being substituted in the fifth, will give the necessary identical relation.

Hence, it follows that the plane represented by the first two equations has, in general, no common points with the line represented by the last three. They may, however, have one common point, under a certain condition.

Hence, *a line and a plane in the fourfold do not, in general, intersect, but they can do so under a certain condition.*

Note.—If the line and the plane have two common points, an additional condition will have to be satisfied and in this case the line will lie entirely in the plane.

(2) Two sets of three linear equations represent two lines. The six equations involving four variables can never be satisfied by one common set of values of the variables, unless two identical relations are satisfied by the co-efficients in the equations. Hence, the two lines represented by these six equations will never intersect.

Thus, *any two lines in the fourfold never intersect, but they can do so under certain conditions.*

22. Summary of the above Results:

The above results on the intersections of linear manifolds in the fourfold may be summarised as follow:

(1) Two hyperplanes intersect in a plane, three hyperplanes intersect in a line and four hyperplanes intersect in a point.

(2) A hyperplane intersects a plane in a line.

(3) A hyperplane intersects a line in only one point.

(4) A plane intersects another plane in only one point.

(5) A plane never intersects a line, unless a certain condition is satisfied.

(6) Two lines can never intersect, unless two conditions are satisfied.

23. Determinative Parameters:

A linear equation $A=0$ defining a hyperplane contains four independent constants, which may be uniquely determined by the conditions that the hyperplane passes through four independent points. Four independent conditions are then sufficient to determine the equation uniquely. Hence, a hyperplane is determined by four conditions.

A plane is determined by three given points. Consider a hyperplane defined by the linear equation $A=0$, which passes through three given points. Then the four parameters in the equation $A=0$ are connected by three linear relations, namely, those obtained by substituting the co-ordinates of the three given points in the equation $A=0$. Eliminating three of the parameters by means of these relations, there remains only one parameter in the equation, which may now be written in the form $B+\lambda C=0$, in which both B and C are linear. The common points of all the hyperplanes of the pencil $B+\lambda C=0$ lie in the plane $B=0$, $C=0$, which, therefore, passes through the three given points.

The two equations $B=0$ and $C=0$ involve eight parameters and all linear combinations of these two equations determine the same plane.

Consider the two equations:—

$$B+\lambda C=0 \quad \text{and} \quad B+\mu C=0.$$

These contain two arbitrary parameters λ and μ, and consequently, the eight parameters, which determine the two hyperplanes separately, are reduced by two and there remain only six parameters for their specification,

i.e., the same six parameters determine the plane common between the two hyperplanes.

Thus, *a plane is determined by six independent parameters*. That a plane is determined by six independent parameters also follows from the following consideration: The plane requires three given points for its specification and these are equivalent to 12 conditions. Each point has two degrees of freedom in the plane implying two conditions. Hence, the number of remaining conditions $= 12 - 3 \times 2 = 6$.

A line is determined by three linear equations implying 12 conditions; but the equation of the net of hyperplanes through the line, *i.e.*, the equation $A + \lambda B + \mu C = 0$, involves 2 arbitrary parameters, λ, μ. Hence, the three hyperplanes through the line involve $2 \times 3 = 6$ constants. Thus, the number of remaining constants is $12 - 6 = 6$, *i.e., a line is determined by six independent parameters.*

A point is evidently determined by four parameters. Thus, a point and a hyperplane are determined by four parameters each, while a plane and a line require six parameters each for their specification.

CHAPTER III

Reference System and Representation

24. Orthogonal Frame of Axes:

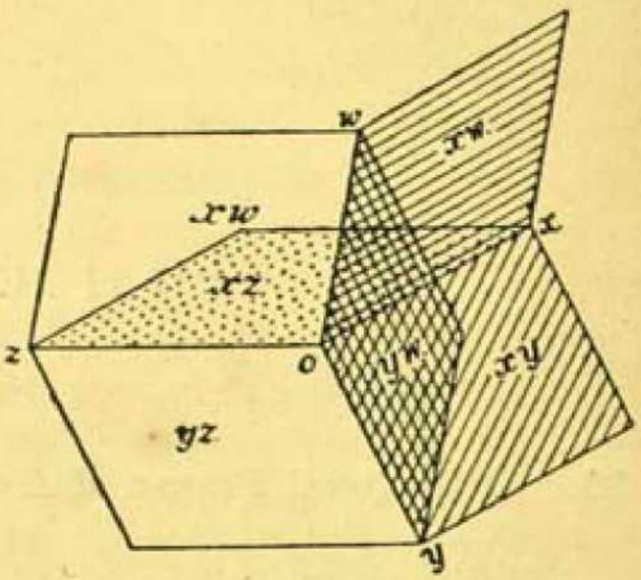
The position of a point in the fourfold may be specified with reference to a system of four axes (mutually orthogonal or otherwise) drawn through a common point O as origin. These four lines OX, OY, OZ, OW are generally designated as the axes of x, y, z and w respectively. As is customary in lower geometries, the directions OX, OY, OZ, OW are regarded as positive, and their opposites, *i.e.*, XO, YO, ZO, WO, are regarded as negative.

Each pair of these four axes determine a plane. There are then six such planes XOY, ZOW ; YOZ, XOW ; ZOX, YOW, which are called the "six co-ordinate planes" of xy, zw ; yz, xw ; zx, yw respectively.

Each group of three of these four axes determine a hyperplane. There are then four such hyperplanes OXYZ, OYZW, OZWX and OWXY, which are called the "four co-ordinate hyperplanes" of xyz, yzw, zwx and wxy respectively.

The position of a point P may then be defined with reference to this orthogonal frame.

From P draw the lines PP_x, PP_y, PP_z and PP_w perpendicular to the four co-ordinate hyperplanes yzw, xzw, xyw and xyz respectively.



It should be noted here that from any external point P, there can be drawn one and only one perpendicular PQ to any hyperplane; for, if any other perpendicular PR could be drawn, then in the triangle PQR, two base-angles at Q and R would be together equal to two right angles, which is untrue in the Euclidian space.

Thus, the perpendiculars PP_x, PP_y, PP_z, PP_w are unique, and we may specify the position of P by means of their lengths, *i.e.*, if x, y, z, w denote the four co-ordinates of P, we have

$$x = PP_x, \quad y = PP_y, \quad z = PP_z, \quad w = PP_w.$$

The position of P may be specified in another way, namely, with reference to the four axes OX, OY, OZ, OW.

Project the line OP on the four axes, M_x, M_y, M_z and M_w denoting respectively the projections of P on these axes, *i.e.*, the feet of the projecting perpendiculars on the axes OX, OY, OZ, OW respectively. Then, the lengths OM_x, OM_y, OM_z, OM_w may be defined to be the co-ordinates of P, *i.e.*,

$$x = OM_x, \quad y = OM_y, \quad z = OM_z, \quad w = OM_w.$$

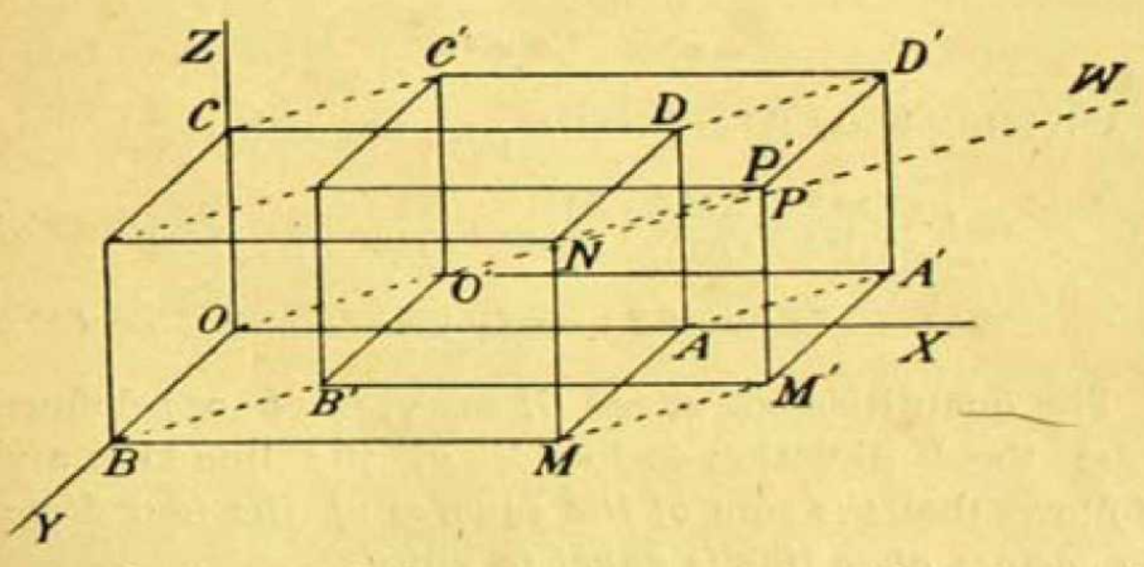
It can be easily shown that $OM_x = PP_x$, $OM_y = PP_y$, $OM_z = PP_z$, $OM_w = PP_w$.

Thus, the position of the point P can be specified with reference to the four co-ordinate axes or the four co-ordinate hyperplanes.

Note.—The four co-ordinates are sometimes denoted by using subscripts such as x_1, x_2, x_3, x_4.

25. A Fundamental Theorem:

Let P be a point (x, y, z, w). If a chain of lines OA, AM, MN and NP′ be drawn from the origin O, respectively parallel to the directions of the co-ordinate axes, and if the successive links of this chain be



respectively equal to x, y, z and w, then the point P′ coincides with P.

The projection of OP on the axis of W is w, and the projection of the chain OAMNP′ on the same axis is also w.

∴ The projection of PP′ on OW = difference of these two projections = 0.

Hence, PP′ is either zero or perpendicular to the axis OW.. But this latter alternative is impossible in

the fourfold, since through any point only four mutually perpendicular lines can be drawn.

$\therefore$ $PP'=0$; *i.e.*, P′ coincides with P.

26. The Direction-cosines of a Line :

Let r denote the length of the line OP and $\alpha, \beta, \gamma, \delta$ the angles which OP makes with the four co-ordinate axes.

From the geometry of the configuration, we have

$$\begin{aligned} OP^2 &= ON^2 + PN^2 \\ &= OA^2 + AM^2 + MN^2 + PN^2 \\ &= x^2 + y^2 + z^2 + w^2. \end{aligned}$$

Dividing both sides by OP^2, we have also

$$1 = \frac{OA^2}{OP^2} + \frac{AM^2}{OP^2} + \frac{MN^2}{OP^2} + \frac{NP^2}{OP^2} \qquad (26.1$$

$$= \cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta. \qquad (26.2$$

The quantities $\cos\alpha$, $\cos\beta$, $\cos\gamma$, $\cos\delta$ are defined to be the " direction-cosines " of the line OP, and it follows that *the sum of the squares of the four direction-cosines of a line is equal to unity.*

Hence, if l, m, n, p represent the actual direction-cosines of a line, we must have $l^2+m^2+n^2+p^2=1$.

If, however, L, M, N, P are quantities proportional to the direction-cosines, then we have

$$\frac{l}{L} = \frac{m}{M} = \frac{n}{N} = \frac{p}{P} = \frac{\sqrt{(l^2+m^2+n^2+p^2)}}{\sqrt{(L^2+M^2+N^2+P^2)}}$$

$$= \frac{1}{\sqrt{(L^2+M^2+N^2+P^2)}}$$

whence the actual direction-cosines are

$$\frac{L}{\sqrt{(L^2+M^2+N^2+P^2)}}, \quad \frac{M}{\sqrt{(L^2+M^2+N^2+P^2)}},$$

$$\frac{N}{\sqrt{(L^2+M^2+N^2+P^2)}}, \quad \frac{P}{\sqrt{(L^2+M^2+N^2+P^2)}}.$$

27. The Angle between Two Lines:

Let OP and OP$'$ be any two lines whose direction-cosines are respectively $l, \ m, \ n, \ p$ and l', m', n', p'. Let x, y, z, w and x', y', z', w' be the co-ordinates of P and P$'$ respectively.

If θ be the angle between OP and OP$'$, then the projection of OP on OP$'$ is OP $\cos\theta$; and this is again equal to the sum of the projections of OA, AM, MN and NP on OP$'$.

$$\therefore \text{ OP} \cos\theta = \text{OA}.\frac{x'}{\text{OP}'} + \text{AM}.\frac{y'}{\text{OP}'} + \text{MN}.\frac{z'}{\text{OP}'} + \text{NP}.\frac{w'}{\text{OP}'}$$

$$= x.l' + y.m' + z.n' + w.p'.$$

$$\therefore \quad \text{Cos}\ \theta = \frac{x}{\text{OP}}.l' + \frac{y}{\text{OP}}.m' + \frac{z}{\text{OP}}.n' + \frac{w}{\text{OP}}.p'$$

$$= ll' + mm' + nn' + pp'. \tag{27.1}$$

Again, $\text{Sin}^2\ \theta = 1 - \cos^2\theta = 1 - (ll' + mm' + nn' + pp')^2$

$$= (l^2+m^2+n^2+p^2)(l'^2+m'^2+n'^2+p'^2) - (ll'+mm'+nn'+pp')^2$$

$$= (lm'-l'm)^2 + (mn'-m'n)^2 + (nl'-n'l)^2 + (lp'-l'p)^2 + (mp'-m'p)^2 + (np'-n'p)^2$$

$$\therefore \quad \text{Sin}\ \theta = \sqrt{\Sigma(lm'-l'm)^2}. \tag{27.2}$$

This is the analogue of Lagrange's Identity in the ordinary space. Using the subscript notation and denoting the direction-cosines of the lines OP and OP′ respectively by l_r and l'_r $(r=1, 2, 3, 4)$, the above two results may be put into the following forms:

$$\text{Cos}\ \theta = \Sigma\, l_r\, l_r' = l_1 l_1' + l_2 l_2' + l_3 l_3' + l_4 l_4'$$

$$\text{Sin}^2\ \theta = \Sigma\, l_r^2 \Sigma\, l'^2_r - (\Sigma l_r l_r')^2$$

$$= \Sigma \begin{vmatrix} l_r & l_s \\ l'_r & l'_s \end{vmatrix}^2, \ (r=1, 2, 3, 4\ ;\ s=1, 2, 3, 4).$$

Note.—When other than orthogonal systems of co-ordinates are used, the quantities l, m, n, p are defined to be the *direction-ratios* or *direction-constants*, and in this case, the relation $l^2+m^2+n^2+p^2=1$ does not hold.

Cor.: When $\theta = \frac{\pi}{2}$, $\cos\ \theta = ll' + mm' + nn' + pp' = 0$, which is the condition of orthogonality of the lines.

28. Non-conjoint Points:

A group of points is said to be *conjoint*, if all its members belong to the same linear manifold.

Thus, any two points specify a straight line ; if a third point also lies on this line, the three points are *conjoint*. But if the third point does not lie on the same line, but specifies a plane together with the other two points, then the three points are said to be *non-conjoint*, or, *independent*.

In a similar manner, if a fourth point lies in the plane specified by any three points, the four points are

said to be conjoint, and in the contrary case, they are said to be non-conjoint or independent.

Similarly, a group of five points are said to be conjoint, when they belong to the same hyperplane and non-conjoint in the contrary case. Hence, *a group of points may be defined to be conjoint or non-conjoint, according as they all belong to the same linear manifold or not.*

Theorem: *To determine the condition that $r+1$ $(r \leq 4)$ given points in the fourfold may be conjoint or non-conjoint.*

Let x_r, y_r, z_r, w_r $(r=0, 1, 2, 3, 4)$ be the co-ordinates of the $(r+1)$ given points. Consider the following five linear equations:—

$$\left.\begin{array}{l} \lambda_0 + \lambda_1 + \lambda_2 + \ldots\ldots + \lambda_r = 0 \\ \lambda_0 x_0 + \lambda_1 x_1 + \ldots\ldots + \lambda_r x_r = 0 \\ \lambda_0 y_0 + \lambda_1 y_1 + \ldots\ldots + \lambda_r y_r = 0 \\ \lambda_0 z_0 + \lambda_1 z_1 + \ldots\ldots + \lambda_r z_r = 0 \\ \lambda_0 w_0 + \lambda_1 w_1 + \ldots\ldots + \lambda_r w_r = 0 \end{array}\right\} (r \leq 4) \quad (28.1$$

which are obtained by equating to zero the same linear combinations of the points with the parameters λ_0, λ_1, $\lambda_2, \ldots \lambda_r$, subject to the condition $\lambda_0 + \lambda_1 + \ldots + \lambda_r = 0$. These $r+1$ points will be conjoint or not (dependent or independent), according as there are or are not values of the $(r+1)$ parameters λ_0, $\lambda_1 \ldots$, not all zeros, which satisfy all the above five equations.

If $r > 4$, the points are necessarily conjoint, because, denoting the characteristic of the matrix by $c(\leq 5)$,

these may be satisfied by taking certain $r+1-c(\geq 0)$ of the $r+1$ parameters.*

If $r \leq 4$, the characteristic c of the matrix is $\leq r+1$. When $c=r+1$, the $r+1$ given points are non-conjoint (independent), as the matrix cannot be satisfied except by values, all zero, of the parameters. When $c < r+1$, the points are conjoint.

Hence, with the convention that a matrix vanishes, when all its minors of maximum order are zero and is different from zero in other cases,† the above conditions may be stated in the following form :—

Any $r+1 (r \leq 4)$ given points in the fourfold are conjoint or not, according as the matrix

$$\left\| \begin{matrix} 1 & 1 & 1 & \dots\dots & 1 \\ x_0 & x_1 & x_2 & \dots\dots & x_r \\ y_0 & y_1 & y_2 & \dots\dots & y_r \\ z_0 & z_1 & z_2 & \dots\dots & z_r \\ w_0 & w_1 & w_2 & \dots\dots & w_r \end{matrix} \right\| = 0, \text{ or } \neq 0. \qquad (28.2$$

Veronese defines‡ r points to be independent or non-conjoint, when none of them belongs to the space determined by the remaining $r-1$ points.

29. The Content of the finite join of five Points :

The finite configuration formed by five independent points in the fourfold is analogous to the tetrahedron

* Capelli, Istituzioni di analisi algebrica, 3rd Ed., n. 445.
† C. E. Cullis, Matrices and Determinoids, Ch. 1, § 30.
‡ Veronese, Grundzüge der Geometrie, etc. § 15, Def. 1, p. 256.

in the ordinary space. It may be called a "Pentahedroid." The content may be calculated as follows :—*

Let the five points be denoted by a_i, b_i, c_i, d_i, e_i respectively $(i=1, 2, 3, 4)$. Then, denoting the content of the *simplex* of these five points by V, we may write

$$V=\frac{1}{4!}\begin{vmatrix} 1 & a_1 & a_2 & a_3 & a_4 \\ 1 & b_1 & b_2 & b_3 & b_4 \\ 1 & c_1 & c_2 & c_3 & c_4 \\ 1 & d_1 & d_2 & d_3 & d_4 \\ 1 & e_1 & e_2 & e_3 & e_4 \end{vmatrix}$$

in the form—

$$(-1)^5 V=\frac{1}{4!}\begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & a_1 & a_2 & a_3 & a_4 & \Sigma a_i^2 \\ 1 & b_1 & b_2 & b_3 & b_4 & \Sigma b_i^2 \\ 1 & c_1 & c_2 & c_3 & c_4 & \Sigma c_i^2 \\ 1 & d_1 & d_2 & d_3 & d_4 & \Sigma d_i^2 \\ 1 & e_1 & e_2 & e_3 & e_4 & \Sigma e_i^2 \end{vmatrix}$$

Then, the product of the two determinants, namely,

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \Sigma a_i^2 & -2a_1 & -2a_2 & -2a_3 & -2a_4 & 1 \\ \Sigma b_i^2 & -2b_1 & -2b_2 & -2b_3 & -2b_4 & 1 \\ \Sigma c_i^2 & -2c_1 & -2c_2 & -2c_3 & -2c_4 & 1 \\ \Sigma d_i^2 & -2d_1 & -2d_2 & -2d_3 & -2d_4 & 1 \\ \Sigma e_i^2 & -2e_1 & -2e_2 & -2e_3 & -2e_4 & 1 \end{vmatrix}$$

* W. J. C. Sharp, Proc. London Math. Soc., Vol. XVIII.

and
$$\begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & a_1 & a_2 & a_3 & a_4 & \Sigma a_i{}^2 \\ 1 & b_1 & b_2 & b_3 & b_4 & \Sigma b_i{}^2 \\ 1 & c_1 & c_2 & c_3 & c_4 & \Sigma c_i{}^2 \\ 1 & d_1 & d_2 & d_3 & d_4 & \Sigma d_i{}^2 \\ 1 & e_1 & e_2 & e_3 & e_4 & \Sigma e_i{}^2 \end{vmatrix}$$

$$=(2)^4(4\,!V)\times(-1)^5(4\,!V)=(-1)^5\cdot2^4.\,(4\,!V)^2$$

$$=\begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & (ab)^2 & (ac)^2 & (ad)^2 & (ae)^2 \\ 1 & (ba)^2 & 0 & (bc)^2 & (bd)^2 & (be)^2 \\ 1 & (ca)^2 & (cb)^2 & 0 & (cd)^2 & (ce)^2 \\ 1 & (da)^2 & (db)^2 & (dc)^2 & 0 & (de)^2 \\ 1 & (ea)^2 & (eb)^2 & (ec)^2 & (ed)^2 & 0 \end{vmatrix} \qquad (29.1$$

where $(ab)^2 \equiv (ba)^2$ = square of the distance between the points a_i and b_i ; and so on.

For a regular pentahedroid, the edges are all equal, and $(ab)^2=(bc)^2=(de)^2=$ etc. $\equiv l$ (say). The above determinant then reduces, on subtracting the last column from each of the others except the first, to

$$\begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & -l^2 & 0 & 0 & 0 & l^2 \\ 1 & 0 & -l^2 & 0 & 0 & l^2 \\ 1 & 0 & 0 & -l^2 & 0 & l^2 \\ 1 & 0 & 0 & 0 & -l^2 & l^2 \\ 1 & l^2 & l^2 & l^2 & l^2 & 0 \end{vmatrix} = (-1)^5 \begin{vmatrix} 1 & -l^2 & 0 & 0 & 0 \\ 1 & 0 & -l^2 & 0 & 0 \\ 1 & 0 & 0 & -l^2 & 0 \\ 1 & 0 & 0 & 0 & -l^2 \\ 1 & l^2 & l^2 & l^2 & l^2 \end{vmatrix}$$

$$=(-1)^5.l^8. \begin{vmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 5 & 0 & 0 & 0 & 0 \end{vmatrix} =(-1)^5.5l^8.$$

Hence, the content of a regular pentahedroid of edge l is given by

$$(-1)^5.2^4.\,(4\,!\,V)^2=(-1)^5.5.l^8$$

or
$$V=\frac{l^4}{4\,!}\sqrt{\frac{5}{2^4}}=\frac{l^4}{2^2.4\,!}\sqrt{5} \qquad (29.2$$

30. Identical relation between mutual distances of six points:

Since any five non-conjoint points determine the fourfold, any sixth point situated in the same fourfold cannot, therefore, be independent ; the six points must necessarily be connected by a certain relation.

Denoting the six points by a_i, b_i, c_i, d_i, e_i, f_i, respectively ($i=1, 2, 3, 4,$), the product of the two following determinoids of 6 columns and 7 rows must give an identically vanishing result:—*

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \Sigma a_i^2 & -2a_1 & -2a_2 & -2a_3 & -2a_4 & 1 \\ \Sigma b_i^2 & -2b_1 & -2b_2 & -2b_3 & -2b_4 & 1 \\ \Sigma c_i^2 & -2c_1 & -2c_2 & -2c_3 & -2c_4 & 1 \\ \Sigma d_i^2 & -2d_1 & -2d_2 & -2d_3 & -2d_4 & 1 \\ \Sigma e_i^2 & -2e_1 & -2e_2 & -2e_3 & -2e_4 & 1 \\ \Sigma f_i^2 & -2f_1 & -2f_2 & -2f_3 & -2f_4 & 1 \end{vmatrix} \times$$

* Burnside and Panton, Theory of Equations, Vol. II, § 143.

$$\times \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & a_1 & a_2 & a_3 & a_4 & \Sigma a_i{}^2 \\ 1 & b_1 & b_2 & b_3 & b_4 & \Sigma b_i{}^2 \\ 1 & c_1 & c_2 & c_3 & c_4 & \Sigma c_i{}^2 \\ 1 & d_1 & d_2 & d_3 & d_4 & \Sigma d_i{}^2 \\ 1 & e_1 & e_2 & e_3 & e_4 & \Sigma e_i{}^2 \\ 1 & f_1 & f_2 & f_3 & f_4 & \Sigma f_i{}^2 \end{vmatrix} = 0$$

whence

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & (ab)^2 & (ac)^2 & (ad)^2 & (ae)^2 & (af)^2 \\ 1 & (ba)^2 & 0 & (bc)^2 & (bd)^2 & (be)^2 & (bf)^2 \\ 1 & (ca)^2 & (cb)^2 & 0 & (cd)^2 & (ce)^2 & (cf)^2 \\ 1 & (da)^2 & (db)^2 & (dc)^2 & 0 & (de)^2 & (df)^2 \\ 1 & (ea)^2 & (eb)^2 & (ec)^2 & (ed)^2 & 0 & (ef)^2 \\ 1 & (fa)^2 & (fb)^2 & (fc)^2 & (fd)^2 & (fe)^2 & 0 \end{vmatrix} = 0 \quad (30.1$$

where $(ab)^2 \equiv (ba)^2 =$ the square of the distance between the points a_i, b_i; and so on.

This is the identical relation satisfied by the mutual distances between any six points in the fourfold.

31. Analytical representation of a straight Line :

The position of a straight line may be specified in different ways ; for instance, a point on the line and its direction, or any two points on the line will uniquely

define its position. A straight line can, therefore, be analytically represented as follows :—

(1) Let $Q(a, b, c, d)$ be a fixed point on the line whose direction-cosines are l, m, n, p.

If, then, $P(x, y, z, w)$ be a variable point on the line, at a distance r from Q, we have

$$x-a=lr, \quad y-b=mr, \quad z-c=nr, \quad w-d=pr.$$

Consequently,

$$(x-a)^2+(y-b)^2+(z-c)^2+(w-d)^2$$
$$=r^2(l^2+m^2+n^2+p^2)=r^2 \tag{31.1}$$

whence

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}=\frac{w-d}{p}=r. \tag{31.2}$$

These are the equations of the line PQ, drawn in the direction determined by l, m, n, p, through the point $Q(a, b, c, d)$.

If L, M, N, P are given quantities proportional to the direction-cosines of the line, the above equations admit of being written down in the forms

$$\frac{x-a}{L}=\frac{y-b}{M}=\frac{z-c}{N}=\frac{w-d}{P}. \tag{31.3}$$

These are, then, the general equations of a line, where L, M, N, P are any quantities whatever.

The co-ordinates of any variable point on the line PQ may then be taken as $a+lr$, $b+mr$, $c+nr$, $d+pr$, where r is the distance of the point from the fixed point (a, b, c, d).

The co-ordinates may also be written as

$$x=a+\mathrm{LR},\ y=b+\mathrm{MR},\ z=c+\mathrm{NR},\ w=d+\mathrm{PR},$$

where L, M, N, P are quantities proportional to the direction-cosines and the distance of the variable point from the fixed point is given by

$$(\mathrm{L}^2+\mathrm{M}^2+\mathrm{N}^2+\mathrm{P}^2)^{\frac{1}{2}}\mathrm{R}.$$

It is easily seen that the above equations of the line are really equivalent to three independent linear equations, and these are then necessary and sufficient for the mathematical specification of the line.

(2) If R (a', b', c', d') be any other point on the line PQ, these co-ordinates must satisfy the equations of the line, and we must have

$$\frac{a'-a}{l}=\frac{b'-b}{m}=\frac{c'-c}{n}=\frac{d'-d}{p}.$$

Hence, eliminating l, m, n, p with the help of equations (31.2), we obtain

$$\frac{x-a}{a'-a}=\frac{y-b}{b'-b}=\frac{z-c}{c'-c}=\frac{w-d}{d'-d}=\lambda \text{ (say)}. \qquad (31.4$$

These are, then, the equations of a line through the two given points Q(a, b, c, d) and R(a', b', c', d').

The co-ordinates of any variable point on this line can, therefore, be taken as

$$x=a+\lambda(a'-a),$$

$$y=b+\lambda(b'-b),$$

$$z=c+\lambda(c'-c),$$

$$w=d+\lambda(d'-d).$$

where λ is a parameter admitting values ranging from

$$-\infty \quad \text{to} \quad +\infty.$$

These clearly show that the co-ordinates may also be written in the forms

$$x=\frac{\lambda a'+\mu a}{\lambda+\mu}, \quad y=\frac{\lambda b'+\mu b}{\lambda+\mu}, \quad z=\frac{\lambda c'+\mu c}{\lambda+\mu}, \quad w=\frac{\lambda d'+\mu d}{\lambda+\mu},$$

$$\text{where } \lambda+\mu=1. \tag{31.5}$$

These agree with the results in the ordinary geometry of two or three dimensions, and it is evident that the co-ordinates are linear functions of the corresponding co-ordinates of the two given points.

If (a'', b'', c'', d'') be a third point on the line QR, *i.e.*, on the join of the two points (a, b, c, d) and (a', b', c', d'), we have

$$(\lambda+\mu)a''=\lambda a'+\mu a, \qquad (\lambda+\mu)b''=\lambda b'+\mu b.$$

$$(\lambda+\mu)c''=\lambda c'+\mu c, \qquad (\lambda+\mu)d''=\lambda d'+\mu d,$$

subject to the condition $\lambda+\mu=1$.

Eliminating λ and μ between these relations, we obtain the following conditions for the collinearity of the three points (a, b, c, d), (a', b', c', d') and (a'', b'', c'', d'') in the form of a matrix equation :

$$\left\| \begin{matrix} a & b & c & d \\ a' & b' & c' & d' \\ a'' & b'' & c'' & d'' \end{matrix} \right\| =0. \tag{31.6}$$

32. Parametric representation of a Plane :

A plane may be specified by any three non-collinear points, or any two lines intersecting at a point.

(1) Let A(a, b, c, d), B(a', b', c', d') and C(a'', b'', c'', d'') be three given non-collinear points.

From what has been said about two distinct linear aggregates determining a plane, and from the characteristic property of a linear manifold (§ 13), it follows that a plane consists of all points collinear with any two of the three points A, B, C and all points collinear with any two points thus obtained.

Hence, if P be any point on the line AB and Q any other point on AC, then, any point R on the line PQ also lies in the plane determined by A, B, C.

The co-ordinates of P may be taken as—

$$x=a+\lambda(a'-a), \qquad y=b+\lambda(b'-b),$$

$$z=c+\lambda(c'-c), \qquad w=d+\lambda(d'-d)$$

and the co-ordinates of Q may be taken as—

$$x=a+\mu(a''-a), \qquad y=b+\mu(b''-b),$$

$$z=c+\mu(c''-c), \qquad w=d+\mu(d''-d),$$

where λ, μ are two independent parameters, admitting values ranging from $-\infty$ to $+\infty$.

Hence, the co-ordinates of R may be written as—

$$x=a+\lambda(a'-a)+\nu[\{a+\mu(a''-a)\}-\{a+\lambda(a'-a)\}]$$

$$=a+\lambda'(a'-a)+\mu'(a''-a), \qquad (32.1$$

where ν is a third parameter, different from λ and μ, and $\lambda'=\lambda-\lambda\nu$, $\mu'=\mu\nu$.

Similarly,

$$\left.\begin{aligned} y &= b + \lambda'(b'-b) + \mu'(b''-b) \\ z &= c + \lambda'(c'-c) + \mu'(c''-c) \\ w &= d + \lambda'(d'-d) + \mu'(d''-d) \end{aligned}\right\} \qquad (32.1$$

These show that the co-ordinates of any point in the plane, determined by any three non-collinear given points, are linear functions of the corresponding co-ordinates of these latter points. These then give a parametric representation of the plane.

If now the parameters λ' and μ' be eliminated between these four relations, we obtain the locus of the point R, *i.e.*, the plane ABC defined by the matrix equations

$$\left\| \begin{matrix} x-a & y-b & z-c & w-d \\ a'-a & b'-b & c'-c & d'-d \\ a''-a & b''-b & c''-c & d''-d \end{matrix} \right\| = 0. \qquad (32.2$$

which are equivalent to two independent linear equations in four variables x, y, z, w, and therefore, represent a plane (§ 15).

(2) Let l, m, n, p be the direction-cosines of the line AB and l', m', n', p' those of the line AC. Then we have—

$$\left.\begin{aligned} \frac{a'-a}{l} &= \frac{b'-b}{m} = \frac{c'-c}{n} = \frac{d'-d}{p} = \alpha \\ \frac{a''-a}{l'} &= \frac{b''-b}{m'} = \frac{c''-c}{n'} = \frac{d''-d}{p'} = \beta \end{aligned}\right\} \qquad (32.3$$

Substituting these values in the equations (32.2), the equations of the plane ABC may also be written in the form of a matrix equation :

$$\left\| \begin{matrix} x-a & y-b & z-c & w-d \\ l & m & n & p \\ l' & m' & n' & p' \end{matrix} \right\| = 0. \qquad (32.4$$

These are the equations of the plane in terms of the co-ordinates of any point A (a, b, c, d) and the direction-cosines of two lines drawn through A and lying in the plane.

Again, in virtue of relations (32.3), we may write the co-ordinates of any point in the plane in the forms

$$\begin{aligned} x &= a + \alpha l + \beta l' \qquad & y &= b + \alpha m + \beta m' \\ z &= c + \alpha n + \beta n' \qquad & w &= d + \alpha p + \beta p' \end{aligned} \qquad (32.5$$

If, therefore, (l'', m'', n'', p'') be the direction-cosines of any line in the plane drawn through the point A (a, b, c, d), its equations are—

$$\frac{x-a}{l''} = \frac{y-b}{m''} = \frac{z-c}{n''} = \frac{w-d}{p''} = -\gamma \text{ (say)}.$$

Hence, the relations (32.5) may be written as—

$$\begin{aligned} \alpha l + \beta l' + \gamma l'' &= 0 \qquad & \alpha m + \beta m' + \gamma m'' &= 0 \\ \alpha n + \beta n' + \gamma n'' &= 0 \qquad & \alpha p + \beta p' + \gamma p'' &= 0 \end{aligned}$$

where α, β, γ are any parameters. (32.6

whence it follows that the direction-cosines of any line through the intersection of any two lines in a plane are linear functions of those of the latter.

Eliminating α, β, γ, we get

$$\left\| \begin{matrix} l & m & n & p \\ l' & m' & n' & p' \\ l'' & m'' & n'' & p'' \end{matrix} \right\| = 0 \qquad (32.7$$

showing that the direction-cosines of any three concurrent lines in a plane satisfy the matrix equations (32.7).

(3) The equations of the plane ABC may be deduced in the forms (32.2) also from the following considerations:

Let $a_1x+b_1y+c_1z+d_1w+e_1=0$ be the equation of a hyperplane through the three given points A, B, C. Then, the co-ordinates of these points must satisfy the equation of the hyperplane, and we must have

$$a_1a+b_1b+c_1c+d_1d+e_1=0$$
$$a_1a'+b_1b'+c_1c'+d_1d'+e_1=0$$
$$a_1a''+b_1b''+c_1c''+d_1d''+e_1=0$$

Now, eliminating the five unknown parameters a_1, b_1, c_1, d_1, e_1 between these four relations, we obtain the matrix equation

$$\left\| \begin{matrix} x & y & z & w & 1 \\ a & b & c & d & 1 \\ a' & b' & c' & d' & 1 \\ a'' & b'' & c'' & d'' & 1 \end{matrix} \right\| = 0. \qquad (32.8$$

which easily reduces to the form (32.2) and is equivalent to two independent linear equations defining the plane ABC.

Cor.: If a fourth point $D(a''', b''', c''', d''')$ lies in the plane (32.2), its co-ordinates must satisfy the equations, and we must have

$$\left\| \begin{matrix} a'-a & b'-b & c'-c & d'-d \\ a''-a & b''-b & c''-c & d''-d \\ a'''-a & b'''-b & c'''-c & d'''-d \end{matrix} \right\| = 0$$

which may also be written in the form—

$$\left\| \begin{matrix} a & b & c & d & 1 \\ a' & b' & c' & d' & 1 \\ a'' & b'' & c'' & d'' & 1 \\ a''' & b''' & c''' & d''' & 1 \end{matrix} \right\| = 0. \qquad (32.9$$

These are the conditions which must be satisfied by the co-ordinates of any four co-planar points.

33. The Orientation-cosines of a Plane:

Let two lines OP (l_1, l_2, l_3, l_4) and OQ (l'_1, l'_2, l'_3, l'_4), drawn through the origin O, define a plane. If $OP=r_1$, and $OQ=r_2$ and $\angle POQ=\omega$, then the co-ordinates of P and Q are respectively

$$(l_1r_1, l_2r_1, l_3r_1, l_4r_1) \quad \text{and} \quad (l'_1r_2, l'_2r_2, l'_3r_2, l'_4r_2).$$

The area Δ of the triangle $POQ=\frac{1}{2}r_1r_2 \sin\omega$ and the area Δ_{xy} of its projection $P'OQ'$ on the co-ordinate plane xy is given by

$$\Delta_{xy}=\tfrac{1}{2}\begin{vmatrix} 1 & l_1 r_1 & l_2 r_1 \\ 1 & l'_1 r_2 & l'_2 r_2 \\ 1 & 0 & 0 \end{vmatrix}=\tfrac{1}{2}r_1 r_2\begin{vmatrix} l_1 & l_2 \\ l'_1 & l'_2 \end{vmatrix}$$

$$=\tfrac{1}{2}r_1 r_2(l_1 l'_2 - l_2 l'_1)$$

$\therefore$ The area $P'OQ'$: area $POQ = \Delta_{xy} : \Delta$

$$=(l_1 l'_2 - l_2 l'_1)/\sin\omega$$

or, $$\Delta_{xy}=\Delta.\frac{l_1 l'_2 - l_2 l'_1}{\sin\omega}=\Delta.\frac{L_{12}}{\sin\omega} \text{ (say).} \qquad (33.1$$

If the triangle POQ be projected on the six co-ordinate planes, the areas of the projected triangles will similarly be obtained by multiplying the area Δ respectively by the quantities—

$$\frac{L_{23}}{\sin\omega},\ \frac{L_{13}}{\sin\omega},\ \frac{L_{12}}{\sin\omega},\ \frac{L_{14}}{\sin\omega},\ \frac{L_{24}}{\sin\omega},\ \frac{L_{34}}{\sin\omega} \qquad (33.2$$

where L_{12}, L_{23}, etc., are the several determinants of the matrix

$$\begin{Vmatrix} l_1 & l_2 & l_3 & l_4 \\ l'_1 & l'_2 & l'_3 & l'_4 \end{Vmatrix}$$

If the six quantities (33.2) be respectively denoted by a, b, c, f, g, h, we at once see that

$$a^2+b^2+c^2+f^2+g^2+h^2=$$

$$\frac{L^2_{23}+L^2_{13}+L^2_{12}+L^2_{14}\ L^2_{24}+L^2_{34}}{\sin^2\omega}$$

$$=\frac{\sin^2\omega}{\sin^2\omega}=1 \quad (27.2). \qquad (33.3$$

From (33.1) it at once follows that $\Delta^2 = \Sigma \Delta^2_{xv}$, or in other words, *the square of the area of any plane triangle is equal to the sum of the squares of its projections on the six co-ordinate planes.*

Again, by considering the identically vanishing determinant

$$\begin{vmatrix} l_1 & l_2 & l_3 & l_4 \\ l'_1 & l'_2 & l'_3 & l'_4 \\ l_1 & l_2 & l_3 & l_4 \\ l'_1 & l'_2 & l'_3 & l'_4 \end{vmatrix} = 0$$

we deduce the relation—

$$L_{12}.L_{34} + L_{23}.L_{14} + L_{13}.L_{24} = 0$$

$$i.e., \quad af + bg + ch = 0. \qquad (33.4$$

Thus, the six quantities a, b, c, f, g, h are connected by the two relations (33.3) and (33.4), and are consequently equivalent to four independent parameters.

These six quantities are called the six *Orientation-cosines* of the plane determined by the lines OP and OQ. If they are multiplied by a constant k, then ka, kb, kc, kf, kg, kh may be called the "six co-ordinates of the plane." These are analogous to the six Plückerian co-ordinates of a line in the ordinary geometry of three dimensions.*

* Salmon, A Treatise on the Analytical Geometry of Three Dimensions, § 58.

Note.—Since the six quantities are equivalent to four independent parameters, they are sufficient to specify the position of a plane through the origin, or through any given point. In fact, they are sufficient to indicate the orientation of any plane without specifying its position uniquely, since six independent parameters are required for unique specification of its position. Hence, by analogy to the direction-cosines of a line, the name "orientation-cosines" has been applied to them. The justification for the word 'cosine' will appear in the sequel.

34. Analytical Representation of a Hyperplane:

Any four non-conjoint points are sufficient to determine a hyperplane uniquely.

Let A (a, b, c, d), B (a', b', c', d'), C (a'', b'', c'', d'') and D (a''', b''', c''', d''') be any four given non-conjoint points.

If, then, $lx+my+nz+pw+k=0$ be the equation of the required hyperplane, the co-ordinates of the four given points must satisfy this equation, and we obtain the following equations of condition :—

$$la+mb+nc+pd+k=0$$

$$la'+mb'+nc'+pd'+k=0$$

$$la''+mb''+nc''+pd''+k=0$$

$$la'''+mb'''+nc'''+pd'''+k=0$$

If, then, the five unknown quantities l, m, n, p, k be eliminated by combining these five relations, we

obtain the equation of the hyperplane in the form of the determinant equation—

$$\begin{vmatrix} x & y & z & w & 1 \\ a & b & c & d & 1 \\ a' & b' & c' & d' & 1 \\ a'' & b'' & c'' & d'' & 1 \\ a''' & b''' & c''' & d''' & 1 \end{vmatrix} = 0 \qquad (34.1$$

This may again be written in the following equivalent form :

$$\begin{vmatrix} x-a & y-b & z-c & w-d \\ a'-a & b'-b & c'-c & d'-d \\ a''-a & b''-b & c''-c & d''-d \\ a'''-a & b'''-b & c'''-c & d'''-d \end{vmatrix} = 0 \qquad (34.2$$

It is easily seen, therefore, that the equation of a hyperplane, determined by three independent lines whose direction-cosines are respectively (l, m, n, p), (l', m', n', p'), (l'', m'', n'', p'') and which are concurrent in the point (a, b, c, d), may be put into the form

$$\begin{vmatrix} x-a & y-b & z-c & w-d \\ l & m & n & p \\ l' & m' & n' & p' \\ l'' & m'' & n'' & p'' \end{vmatrix} = 0 \qquad (34.3$$

35. Parametric Representation :

From what has been stated (§ 13), it is clear that a hyperplane may be considered as consisting of all points collinear with any two of the four given points

A, B, C, D and all points collinear with any two obtained in this way.

As before, the co-ordinates of any point P in the plane ABC may be expressed as—

$$x=a+\lambda a'+\mu a'' \qquad z=c+\lambda c'+\mu c''$$

$$y=b+\lambda b'+\mu b'' \qquad w=d+\lambda d'+\mu d'',$$

where λ and μ are two independent parameters. Again, the co-ordinates of any point Q on the line AD may be taken as—

$$x=a+\nu a''',\ y=b+\nu b''',\ z=c+\nu c''',\ w=d+\nu d''',$$

where ν is a third parameter.

Hence, the co-ordinates of any point R on the line PQ are of the forms—

$$\left.\begin{aligned}
x&=a+\lambda a'+\mu a''+\rho(a+\nu a''')=(1+\rho)a+\lambda a'+\mu a''+\rho\nu a'''\\
y&=b+\lambda b'+\mu b''+\rho(b+\nu b''')=(1+\rho)b+\lambda b'+\mu b''+\rho\nu b'''\\
z&=c+\lambda c'+\mu c''+\rho(c+\nu c''')=(1+\rho)c+\lambda c'+\mu c''+\rho\nu c'''\\
w&=d+\lambda d'+\mu d''+\rho(d+\nu d''')=(1+\rho)d+\lambda d'+\mu d''+\rho\nu d'''
\end{aligned}\right\} \quad (35.1)$$

where ρ is a fourth independent parameter, showing that the co-ordinates of any point in the hyperplane, *i.e.*, the linear manifold of three dimensions, may be expressed as linear functions of the corresponding co-ordinates of the four points defining the hyperplane. Since λ, μ, ν, ρ are all independent parameters, each admitting values which range from $-\infty$ to $+\infty$, by eliminating these parameters, the equation of the hyperplane may be obtained in the form (34.1).

36. Guiding Lines of a Hyperplane :

A hyperplane may be specified also by any three non-coplanar, but concurrent, lines. The three lines will be determined by their common point and one point on each of the lines, thus making up four points of the hyperplane. The three lines may be called the "guiding lines" of the hyperplane.

$$\text{Let} \quad Ax+By+Cz+Dw+E=0 \qquad \dots \qquad (1)$$

be the equation of the hyperplane determined by the three lines through the point (a, b, c, d) whose direction-cosines are (l_r, m_r, n_r, p_r) $(r=1, 2, 3)$.

Then, the co-ordinates of three points, one on each of the lines, may be taken as—

$$a+l_rk_r,\ b+m_rk_r,\ c+n_rk_r,\ d+p_rk_r \quad (r=1, 2, 3)$$

Substituting these values in the assumed equation (1) of the hyperplane, we obtain

$$A(a+l_rk_r)+B(b+m_rk_r)+C(c+n_rk_r)$$
$$+D(d+p_rk_r)+E=0 \quad \dots \quad (2)$$
$$(r=1, 2, 3)$$

Also, since (a, b, c, d) is a point in the hyperplane, we have

$$Aa+Bb+Cc+Dd+E=0 \qquad \dots \qquad (3)$$

Eliminating the five unknown parameters A, B, C, D, E with the help of the five equations (1), (2), (3), we

obtain the required equation of the hyperplane in the determinant form—

$$\begin{vmatrix} x-a & y-b & z-c & w-d \\ l_1 & m_1 & n_1 & p_1 \\ l_2 & m_2 & n_2 & p_2 \\ l_3 & m_3 & n_3 & p_3 \end{vmatrix} = 0 \tag{36.1}$$

Let (l, m, n, p) be the direction-cosines of a fourth line through (a, b, c, d), lying in the same hyperplane. Its equations are then

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} = \frac{w-d}{p}$$

Hence, from (36.1) it follows immediately that

$$\begin{vmatrix} l & m & n & p \\ l_1 & m_1 & n_1 & p_1 \\ l_2 & m_2 & n_2 & p_2 \\ l_3 & m_3 & n_3 & p_3 \end{vmatrix} = 0 \tag{36.2}$$

This shows that three parameters λ, μ, ν can be so determined that

$$l = \lambda l_1 + \mu l_2 + \nu l_3, \qquad n = \lambda n_1 + \mu n_2 + \nu n_3,$$

$$m = \lambda m_1 + \mu m_2 + \nu m_3, \qquad p = \lambda p_1 + \mu p_2 + \nu p_3,$$

i.e., the direction-cosines of any line in a hyperplane determined by three guiding lines are linear functions of the direction-cosines of these lines.

37. Normal to a Hyperplane :

Consider the hyperplane defined by the linear equation

$$lx + my + nz + pw + k = 0 \qquad \dots \qquad (1)$$

If the hyperplane passes through the origin, the constant term k will be zero, and the equation reduces to

$$lx + my + nz + pw = 0 \qquad \dots \qquad (2)$$

Let l_r, m_r, n_r, p_r $(r = 1, 2, 3)$ be the direction-cosines of the guiding lines through the origin.

The equations of these lines are of the forms—

$$\frac{x}{l_r} = \frac{y}{m_r} = \frac{z}{n_r} = \frac{w}{p_r} = k_r \qquad \dots \qquad (3)$$

whence from (2), we obtain

$$ll_r + mm_r + nn_r + pp_r = 0 \qquad \dots \qquad (4)$$

$$(r = 1, 2, 3)$$

These relations at once show that the line, whose direction-cosines are proportional to l, m, n, p, is perpendicular to each of the three guiding lines of the hyperplane (§ 27, Cor.), and consequently, to every line in the hyperplane drawn through the common point, *i.e.*, the origin. Thus, we find that the co-efficients in the equation of a hyperplane are proportional to the direction-cosines of a line, which is perpendicular to every line lying in the hyperplane. This line is called the *normal* to the hyperplane.

Cor. : At any point of a hyperplane, there is then one unique normal direction whose direction-cosines are

given by the co-efficients of the variables in the equation of the hyperplane. Two normals at any two points of the hyperplane will, therefore, be parallel and lie in the same plane.

38. Recapitulation of the Results:

From the above analytical representation of the line, the plane and the hyperplane, the truth of the statements in § 15 will now be manifest.

It is to be noted that—

(1) a hyperplane is represented by one linear equation in four variables and the co-efficients in the equation are proportional to the direction-cosines of the normal to the hyperplane. It is uniquely determined by four given non-conjoint points. If a fifth point is to be situated in the hyperplane determined by any four points, a certain condition must be satisfied, which may be called the *condition of co-spatiality* of five points.

(2) a plane is represented by two linear equations in four variables, and is uniquely determined by three given non-conjoint points. One condition is necessary for a fourth point to lie in the plane, which is called the *condition of co-planarity* of four points. It may be remarked in passing that the normal to the hyperplane defined by each of the two equations is orthogonal to the plane, and consequently, at any point of the plane, two different lines, orthogonal to the plane, can be drawn—a phenomenon peculiar to the fourfold. This is a very interesting fact and will lead to further interesting and important results.

(3) a line is represented by three linear equations in four variables and the normals to the three hyperplanes, defined by these equations, are all perpendicular to the line, *i.e.*, at any point of a line, there are three independent lines perpendicular to the line, and these latter lines determine the hyperplane of which the given line is the normal. The line is determined uniquely by two given points ; for a third point to lie on the line, a condition of collinearity of the three points must be fulfilled.

(4) a point is represented by four linear equations defining four hyperplanes, *i.e.*, four hyperplanes meet in one point. The four normals do not lie in the same hyperplane and are, therefore, four independent lines through the point (§ 13).

CHAPTER IV

INTERSECTION AND PARALLELISM

39. The Case of Two Hyperplanes:

Consider the two hyperplanes represented by the equations

$$A \equiv ax + by + cz + dw + e = 0 \qquad \dots \quad (1)$$

$$B \equiv a'x + b'y + c'z + d'w + e' = 0 \qquad \dots \quad (2)$$

The common points of the two hyperplanes lie on the plane defined by the two equations (1) and (2), and the hyperplanes are said to intersect in the plane. If, however,

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} = \frac{d}{d'},$$

the two equations are inconsistent ; no finite values of the variables can be found which simultaneously satisfy both the equations.

If e/e' is also equal to each of these ratios, having a common value λ, the second equation takes the form

$$B \equiv \lambda A = 0,$$

showing that the two hyperplanes are identical and there is no plane of intersection.

On the other hand, if e/e' has a value different from that of the equal ratios, the second equation takes the form

$$B \equiv \lambda A + \epsilon = 0,$$

where ϵ is a non-vanishing constant. The equation of the second hyperplane can, therefore, be written in the form $B \equiv A+k=0$, where k is also a non-vanishing constant. It is easily seen, then, that no finite values of the variables can be found to satisfy both the equations $A=0$ and $A+k=0$. This inconsistency is explained by the fact that the variables have no finite values, and the points denoted by these infinite values are all situated at an infinite distance on a certain plane, which may be called the *Plane at Infinity*.

In this case, the two hyperplanes are said to be "parallel." They have no finite common points, but intersect in a plane at infinity. Hence, *the condition that any two hyperplanes may be parallel is that the co-efficients of the variables, and not the constant terms, in their equations should be proportional, and the equations differ by a constant only.*

Cor.: Two parallel hyperplanes have the same normal direction.

40. The Case of a Plane and a Hyperplane:

Consider the hyperplane

$$A \equiv a_1x+b_1y+c_1z+d_1w+e_1=0 \qquad \dots \quad (1)$$

and the plane

$$\left.\begin{aligned} B &\equiv a_2x+b_2y+c_2z+d_2w+e_2=0 \\ C &\equiv a_3x+b_3y+c_3z+d_3w+e_3=0 \end{aligned}\right\} \qquad \dots \quad (2)$$

These two intersect in a line defined by the simultaneous equations

$$A=0, \quad B=0, \quad C=0.$$

The plane (2) will lie entirely in the hyperplane (1), if the first hyperplane $A=0$ is one of the hyperplanes of the pencil defined by $B+\lambda C=0$, where λ is a parameter. The condition for this is—

$$\frac{a_2+\lambda a_3}{a_1}=\frac{b_2+\lambda b_3}{b_1}=\frac{c_2+\lambda c_3}{c_1}$$

$$=\frac{d_2+\lambda d_3}{d_1}=\frac{e_2+\lambda e_3}{e_1}=-\alpha(\text{say}).$$

These are equivalent to five relations between the two parameters λ and α.

Eliminating λ and α between them, the conditions are obtained in the form of the matrix equation

$$\left\| \begin{matrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 \end{matrix} \right\| = 0 \qquad (40.1$$

If, however, the last ratio is not equal to $-\alpha$, *i.e.*, if $\frac{e_2+\lambda e_3}{e_1}\neq-\alpha$, then no finite values can be given to the variables which satisfy all the three equations simultaneously, *i.e.*, the hyperplane $A=0$ will be parallel to $B+\lambda C=0$, and we must have

$$\frac{a_2+\lambda a_3}{a_1}=\frac{b_2+\lambda b_3}{b_1}=\frac{c_2+\lambda c_3}{c_1}=\frac{d_2+\lambda d_3}{d_1}=-\alpha$$

but
$$\frac{e_2+\lambda e_3}{e_1}\neq-\alpha$$

i.e.,

$$\alpha a_1 + a_2 + \lambda a_3 = 0$$

$$\alpha b_1 + b_2 + \lambda b_3 = 0$$

$$\alpha c_1 + c_2 + \lambda c_3 = 0$$

$$\alpha d_1 + d_2 + \lambda d_3 = 0.$$

Eliminating λ and α between these four relations, we obtain the necessary conditions in the matrix form—

$$\left\| \begin{matrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{matrix} \right\| = 0 \qquad (40.2$$

Hence, the hyperplane and the plane will intersect in a line at infinity, if conditions (40.2) are satisfied and conditions (40.1) are not satisfied. In the latter case, when (40.1) holds, all the three hyperplanes have a common plane of intersection.

41. A Line and a Hyperplane :

We have already seen that in the fourfold a line always intersects a hyperplane in only one point. If this only point of intersection lies at an infinite distance, the line is said to be *parallel* to the hyperplane.

Let the hyperplane be represented by $A=0$, and the line by the three linear equations

$$B=0, \quad C=0, \quad D=0.$$

The co-ordinates of the point of intersection are obtained by solving these four linear equations, which

give unique values of the variables. If both the numerator and denominator in the values thus obtained are finite, the point is at a finite distance. If the denominator is zero, but not the numerators, the values are infinite and the point lies at an infinite distance. Hence, the line will intersect the hyperplane at a point at infinity, *i.e.*, will be parallel to the hyperplane, if the determinant of the co-efficients vanishes, *i.e.*, if

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0$$

but

$$\begin{Vmatrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 \\ a_4 & b_4 & c_4 & d_4 & e_4 \end{Vmatrix} \neq 0 \qquad (41.1$$

If the line be defined by the equations

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} = \frac{w-d}{p} = r$$

and the hyperplane by

$$a_1x + b_1y + c_1z + d_1w + e_1 = 0,$$

then, the co-ordinates of the point of intersection are obtained from the equation

$$a_1(a+lr) + b_1(b+mr) + c_1(c+nr) + d_1(d+pr) + e_1 = 0$$

or, $r(a_1l + b_1m + c_1n + d_1p) + (aa_1 + bb_1 + cc_1 + dd_1 + e_1) = 0.$

This gives the distance r of the point of intersection from the point (a, b, c, d).

This point will be at infinity, if $r = \infty$,

$$\text{i.e., if} \qquad a_1 l + b_1 m + c_1 n + d_1 p = 0$$

$$\text{and} \qquad aa_1 + bb_1 + cc_1 + dd_1 + c_1 \neq 0 \qquad (41.2$$

which show that the line is at right angles to the normal direction of the hyperplane. If also

$$aa_1 + bb_1 + cc_1 + dd_1 + e_1 = 0$$

the intersection is indeterminate, *i.e.*, if the point (a, b, c, d) lies in the hyperplane and the line is at right angles to the normal direction of the hyperplane, it lies entirely in the hyperplane.

42. The Case of a Line and a Plane :

(1) Consider the line $A=0$, $B=0$, $C=0$ and the plane $D=0$, $E=0$.

There are five equations involving four variables, and consequently no solution exists, *i.e.*, no set of common values can be obtained which will satisfy all the equations, unless a certain condition is satisfied.

This is possible only when the equations are not all independent. Eliminating the four variables from the given equations, we obtain the necessary condition in the form—

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 \\ a_4 & b_4 & c_4 & d_4 & e_4 \\ a_5 & b_5 & c_5 & d_5 & e_5 \end{vmatrix} = 0 \qquad (42.1$$

If this condition is satisfied, the line will intersect the plane in a finite point, and the two will lie in the same hyperplane. If, in addition, the determinants of the co-efficients of any four of the five equations vanish, *i.e.*, if the matrix

$$\left\| \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \end{matrix} \right\| = 0 \qquad (42.2$$

the line intersects the plane not at a finite point, but at a point at an infinite distance. In this case the line is *parallel* to the plane.

If all the minor determinants of the fourth order in the determinant (42.1) vanish, the point of intersection is indeterminate and the line lies entirely in the plane.

(2) If the line is given by

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} = \frac{w-d}{p} = r \qquad \dots \quad (1)$$

and the plane by D=0 and E=0, then

$$a+lr, \quad b+mr, \quad c+nr, \quad d+pr$$

must satisfy both the equations D=0 and E=0, *i.e.*, we must have

$$a_4(a+lr)+b_4(b+mr)+c_4(c+nr)+d_4(d+pr)+e_4=0 \dots (2)$$

$$a_5(a+lr)+b_5(b+mr)+c_5(c+nr)+d_5(d+pr)+e_5=0 \dots (3)$$

Eliminating r between (2) and (3), the required condition is obtained in the form

$$\frac{aa_4+bb_4+cc_4+dd_4+e_4}{aa_5+bb_5+cc_5+dd_5+e_5}=\frac{a_4l+b_4m+c_4n+d_4p}{a_5l+b_5m+c_5n+d_5p}. \quad (42.3$$

If $\qquad a_4l+b_4m+c_4n+d_4p=0$

and $\qquad a_5l+b_5m+c_5n+d_5p=0, \qquad (42.4$

but $\qquad aa_4+bb_4+cc_4+dd_4+e_4 \neq 0$

and $\qquad aa_5+bb_5+cc_5+dd_5+e_5 \neq 0$

the intersection is at an infinite distance, and the line is *parallel* to the plane.

If also the latter two expressions vanish, the line lies entirely in the plane.

(3) The conditions that the line

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}=\frac{w-d}{p}=r \qquad \ldots \quad (4)$$

intersects the plane

$$\left\|\begin{matrix} x-\alpha & y-\beta & z-\gamma & w-\delta \\ l_1 & m_1 & n_1 & p_1 \\ l_2 & m_2 & n_2 & p_2 \end{matrix}\right\|=0 \qquad \ldots \quad (5)$$

are obtained by substituting the values $a+lr$, $b+mr$, $c+nr$, $d+pr$ respectively for x, y, z, w in (5) ; and we obtain

$$\left\|\begin{matrix} a+lr-\alpha & b+mr-\beta & c+nr-\gamma & d+pr-\delta \\ l_1 & m_1 & n_1 & p_1 \\ l_2 & m_2 & n_2 & p_2 \end{matrix}\right\|=0$$

This is equivalent to two equations involving r, and the necessary condition for intersection will be obtained by eliminating r between those two equations.

If, however, these are to be satisfied, whatever be the values of r, we must have the two following matrix equations:

$$\begin{Vmatrix} a-\alpha & b-\beta & c-\gamma & d-\delta \\ l_1 & m_1 & n_1 & p_1 \\ l_2 & m_2 & n_2 & p_2 \end{Vmatrix}=0$$

and

$$\begin{Vmatrix} l & m & n & p \\ l_1 & m_1 & n_1 & p_1 \\ l_2 & m_2 & n_2 & p_2 \end{Vmatrix}=0. \qquad (42.5$$

The first equation shows that the point (a, b, c, d) must lie in the plane, while the second shows that the line lies in the plane of the two lines (l_1, m_1, n_1, p_1), and (l_2, m_2, n_2, p_2) (32.6).

If the last condition is satisfied, and not the first, then the line is *parallel* to the plane.

43. The Case of Two Planes:

Consider the two planes defined respectively by the pairs of equations—

$$\left.\begin{matrix} A=0 \\ B=0 \end{matrix}\right\} \quad \text{and} \quad \left.\begin{matrix} C=0 \\ D=0 \end{matrix}\right\}$$

These four linear equations in four variables have a unique solution, showing that the two planes intersect

in only one point, specified by the common values of the variables.

This point of intersection will be at an infinite distance, if the denominator in the values of the variables is zero, or in other words, if the determinant of the coefficients vanish, *i. e.*, if

$$\Delta_1 \equiv \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0. \qquad (43.1$$

If, however, the numerators also vanish at the same time, the values are indeterminate, and the two planes will intersect in a line, and they will lie in a hyperplane. The required condition is, then,

$$\Delta_2 \equiv \begin{Vmatrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 \\ a_4 & b_4 & c_4 & d_4 & e_4 \end{Vmatrix} = 0. \qquad (43.2$$

It is to be noted that, in this case, the four equations reduce to only three independent equations.

Thus, we find that the two planes will lie in a hyperplane, if $\Delta_2 = 0$. They will intersect at a point at infinity if $\Delta_1 = 0$ and $\Delta_2 \neq 0$. In this latter case, the two planes are said to be *half-parallel*.

Again, the two planes may lie in the same hyperplane and their line of intersection may lie entirely at infinity. In that case, the planes are said to be

full-parallel, or completely *parallel*. This latter case is within our experience in the ordinary space, but the former has no analogue in the same.

Note.—Other conditions are necessary for the two planes being full-parallel.

44. Two Straight Lines:

Consider the lines

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}=\frac{w-d}{p}=r$$

and $$\frac{x-a'}{l'}=\frac{y-b'}{m'}=\frac{z-c'}{n'}=\frac{w-d'}{p'}=r'.$$

They do not, in general, intersect at any point. If they do so, they lie in a plane, and in that case, values of r and r' may be determined so as to satisfy the relations

$$a+lr=a'+l'r', \qquad b+mr=b'+m'r',$$

$$c+nr=c'+n'r', \qquad d+pr=d'+p'r'. \qquad \dots \quad (1)$$

Eliminating r and r' between these equations, we obtain

$$\left\| \begin{matrix} l & m & n & p \\ l' & m' & n' & p' \\ a-a' & b-b' & c-c' & d-d' \end{matrix} \right\| = 0. \qquad (44.1$$

This is equivalent to two conditions, which must be satisfied in order that the two lines may intersect.

If r and r' are both infinite, it at once follows from (1) that

$$l=l', \quad m=m', \quad n=n', \quad p=p'. \qquad (44.2$$

Hence, the lines will be parallel, if their direction-cosines are the same or proportional, *i.e.*, if they have the same direction.

Using the formula (27.2), we find that the angle θ between two parallel lines is given by

$$\text{Sin } \theta=0, \text{ whence } \theta=0 \text{ or } \pi.$$

45. Nature of Infinity *:

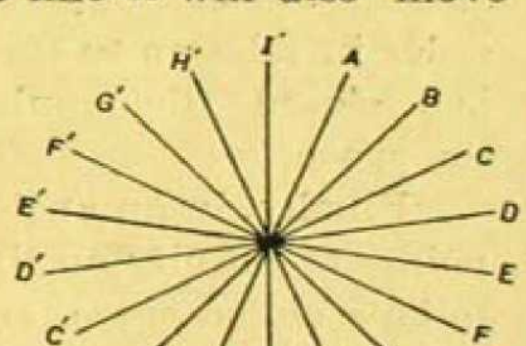
In Mathematics, the word 'infinite' is used in the sense 'limitless,' designating a quantity conceived as constantly increasing or decreasing beyond all limit. Mathematicians further recognise that one infinite quantity may be infinitely great in relation to another infinite quantity, thus conceiving orders of infinity. Infinitely great and infinitely small quantities are regarded as two *poles*, and there is no distinction in their treatment. Introduction of these ideas in geometry has widened the field of geometrical reasoning, leading to the study of geometrical figures, etc., which do not lie in the finite part of the space. Thus, infinity in geometry is regarded as having a distinct geometric existence in the manifold, different from all others of the same category. It is, in fact, unlimited, existing beyond the finite part of the manifold.

Analytical investigation of these geometrical elements at infinity presents considerable difficulties, as the necessary relations between co-ordinates, which are not finite, are somewhat paradoxical. It is, therefore, necessary to examine more closely the infinite region, which has a geometry of its own and where Euclidean postulates are found inapplicable.

* The present exposition proceeds on the lines of E. Jouffret, Traité élémentaire de Géométrie a quatre dimensions, Ch. II, § 7.

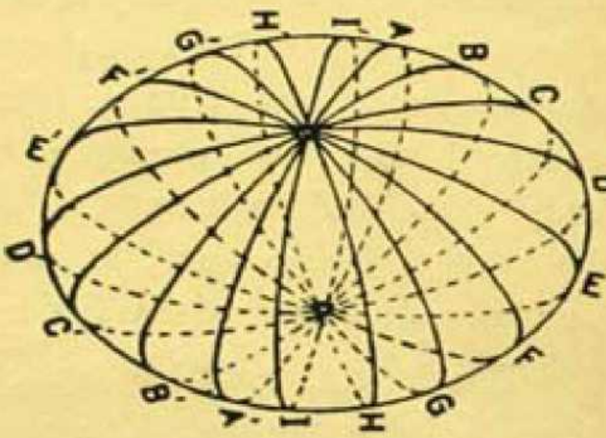
Consider a pencil of rays in a plane $\mathfrak{P}$, issuing from a point O in all possible directions. The locus of the points situated at an infinite distance on each of these rays is a line, not the usually *open* Euclidean line, but a closed Riemannian line, surrounding the plane as the circumference of a circle. This is not to be regarded as an ordinary circle, but it is a circle of the 'first degree.' It is determined by two non-opposite points (and not three). In fact, the two points on each of the two opposite rays are to be counted as only one. This line Ω, surrounding the plane, is to be regarded as a unit, infinite of the first order, relative to the unit nsed in the finite region.

The line Ω may be represented diagrammatically in the adjoining figure. The pencil O is in the finite region of the plane, but the line ABC......A′B′C′......with the opposite points corresponding with the rays A, B, C,.... A′, B′, C′,......is at infinity. If, now, O moves in the plane in all possible manner, the line Ω will also move likewise and the locus of all such positions of Ω is called the *infinite region* of the plane. This locus is evidently of two dimensions like the plane $\mathfrak{P}$, but one of these dimensions, which corresponds to the displacement of Ω, is of the same order of magnitude as the finite displacement of O. Hence, this dimension is infinitely small as compared with the other, and consequently, it reduces to null, and in this sense and in this sense only, we say that the infinite region of the plane is a



closed line, but capable of being replaced by an open one.

Similarly, in the space, the locus of the points at infinity on the rays of a bundle O is a closed or a Riemannian plane $\mathfrak{R}$, which envelopes the space like a sphere, but this latter is not to be regarded as an ordinary sphere. Two opposite rays form a line, which meets this plane $\mathfrak{R}$ in two opposite points, through which pass all the closed rays of the bundle, just as the geographical meridians on a sphere. If



now the bundle O moves in space, the plane $\mathfrak{R}$ will also move, and the locus of this plane is the infinite region of the space, which, therefore, is also of three dimensions ; but one of these is infinitely small of the first order in relation to the other two. This locus, then, is a closed Riemannian plane forming the limit of the space.

Finally, in an analogous manner, the locus of the points at infinity on the rays in the fourfold is found to be a closed space, enveloping it like a hyper-sphere of four dimensions. Strictly speaking, the infinite region of the fourfold is a locus of four dimensions of an infinitude of spaces, corresponding to the different positions of the point O in the fourfold.

Thus, it is seen that the infinitely remote figures in the space help us in conceiving what is beyond the space, and really what is the geometry of the fourfold.

Geometrical infinity, therefore, appears differently in different manifolds. It is a line in the plane, a plane in the space (hyperplane) and a space (hyperplane) in the fourfold. These clearly follow from the linear equations, which analytically represent these manifolds. On the other hand, it is a circumference for the plane, the surface of a sphere for the hyperplane, and the boundary of a hypersphere for the fourfold. Infinity, in fact, has no definite form in any case.*

46. Elements at an Infinite Distance:

From foregoing discussions it appears that in the fourfold there are points, lines, and planes at an infinite distance. A line has a single point at infinity, its intersection with any parallel line. A plane has a line at infinity, its intersection with a parallel plane. A hyperplane has a plane at infinity, its intersection with a parallel hyperplane. It is also clear that the line at infinity of a plane contains all the points at infinity of its lines and the plane at infinity of a hyperplane contains all the points at infinity of its lines and the lines at infinity of its planes. All these points, lines and planes at infinity are all comprised in a single hyperplane, which may be called the *hyperplane at infinity*. For, a hyperplane has a plane at infinity and there are lines which intersect the hyperplane but do not lie in it. These lines have points at infinity which do not

* For further information, the reader may consult Lechalas, Identité des plans de Riemann et des sphères d'Euclide—Mathesis (1898).

G. Veronese, Grundzüge der Geometrie von mehreren dimensionen, etc., translated by Adolf Schepp, Cap. VI, pp. 96-194.

lie in the plane at infinity of this hyperplane. Hence, we conceive that all points, lines and planes at an infinite distance in the fourfold are comprised in a single hyperplane at infinity.* This conception is useful in studying parallelism in the fourfold, as will appear in the sequel.

Theorem: *All points at an infinite distance in the fourfold lie in a single hyperplane.*

If, in the equation

$$A \equiv ax + by + cz + dw + e = 0$$

of any hyperplane, we put

$$x = \infty, \quad y = \infty, \quad z = \infty, \quad w = \infty,$$

then the equation will be satisfied, whatever be the values of the co-efficients, by giving to e the values $\pm\infty$. Hence, any hyperplane whose four points are at an infinite distance has the same hyperplane for its limit, defined by the parameter $e = \infty$, the others being arbitrary.

Thus, it will be seen that the point at infinity of a line, the line at infinity of a plane, the plane at infinity of a hyperplane are nothing but the respective intersections of the line, plane, and hyperplane with this hyperplane at infinity.

47. Linear Elements:

Theorem: *Given two planes in general position. If a third plane intersects them both in lines, it passes through the common point of the two given planes.*

* Elements at infinity are sometimes called *ideal* elements, such as *ideal* points, *ideal* lines, etc.

Let the two planes be defined by

$$A=0, \quad B=0 \quad \text{and} \quad C=0, \quad D=0 \quad \text{respectively.}$$

Then, the generating hyperplane of the first plane may be taken as $A+\lambda B=0$ and that of the second plane as

$$C+\mu D=0,$$

where λ and μ are parameters, each admitting infinite number of values.

Hence, a third plane, which intersects both the given planes in lines, must lie in both the hyperplanes

$$A+\lambda B=0, \qquad C+\mu D=0, \qquad \ldots \quad (1)$$

and these equations may be taken to represent such a plane.

From the forms of these equations, it is evident that they are satisfied by $A=0$, $B=0$, $C=0$, $D=0$, whatever be the values of λ and μ.

Hence, the plane intersecting the two given planes in lines passes through the common point of the two given planes. Since its equations involve two parameters, a doubly infinite system of such planes can be drawn intersecting the two given planes in lines.

Consequently, the two given planes may be regarded as each being generated by a set of lines, one and only one through each point of either plane and all passing through the common point of the two planes. Any two such lines are coplanar. These lines may be called the "linear elements" of the planes.

Let x', y', z', w' be the co-ordinates of a point lying outside of both the planes. If the third plane passes through this point, we must have

$$A' + \lambda B' = 0, \quad C' + \mu D' = 0,$$

where A', B', C', D' are the results of substituting x', y', z', w' respectively for x, y, z, w in A, B, C, D. Eliminating λ and μ between these and the equations (1), we obtain the equations of the third plane in the forms—

$$AB' - A'B = 0 \quad \text{and} \quad CD' - C'D = 0. \qquad (47.1$$

This is then a unique plane, and it follows that *through any point lying outside of both of two given planes passes one and only one plane intersecting them both in lines.*

CHAPTER V

PROJECTION AND ORTHOGONALITY

48. Projection of a Point, Line, etc. :

It can be easily shown that from any given external point P, only one perpendicular can be drawn to a line, a plane or a hyperplane. If Q be the foot of this perpendicular, PQ is the minimum length of all straight lines that can be drawn from P to the line or the plane or the hyperplane. The point Q is called the *projection* of the point P on the line, the plane or the hyperplane, as the case may be.

The projection of a line is defined to be the locus of the feet of the perpendiculars that can be drawn from the different points of the line on any line or plane or hyperplane.

The projection of a plane may similarly be defined as the locus of the feet of the perpendiculars drawn from its points on any plane or hyperplane.

49. Projection of a Point on a Hyperplane :

Let $Q(x, y, z, w)$ be the foot of the perpendicular drawn from any point $P(a, b, c, d)$ on the hyperplane

$$A \equiv a_1x + b_1y + c_1z + d_1w + e_1 = 0. \quad \dots \quad (1)$$

Let $Q'(x+\delta x,\ y+\delta y,\ z+\delta z,\ w+\delta w)$ be another point in the hyperplane very near to Q.

Then, we have—

$$a_1\delta x + b_1\delta y + c_1\delta z + d_1\delta w = 0. \quad \ldots \quad (2)$$

Also, $$PQ^2 = (x-a)^2 + (y-b)^2 + (z-c)^2 + (w-d)^2. \quad \ldots \quad (3)$$

If, then, PQ is to be a minimum, the differential of (3) must vanish, *i.e.*, we must have

$$(x-a)\delta x + (y-b)\delta y + (z-c)\delta z + (w-d)\delta w = 0. \quad \ldots \quad (4)$$

Since the relation (4) will be satisfied when (2) is satisfied, by comparing the two, we obtain, λ being a parameter,

$$x-a=\lambda a_1, \qquad z-c=\lambda c_1,$$

$$y-b=\lambda b_1, \qquad w-d=\lambda d_1,$$

Eliminating λ between these, we deduce the relations

$$\frac{x-a}{a_1} = \frac{y-b}{b_1} = \frac{z-c}{c_1} = \frac{w-d}{d_1} \qquad (49.1$$

showing that PQ is parallel to the normal to the hyperplane at the point Q.

These relations, combined with (1), will determine the co-ordinates of Q.

If, on the other hand, the point Q be fixed and P be allowed to vary, *i.e.*, if x, y, z, w are regarded as constants and a, b, c, d as variables, it is clear that the equations (49.1) determine the locus of all points whose projections on A all coincide with Q. This is then a straight line perpendicular to the hyperplane at the point Q. PQ is thus the perpendicular distance of the point P from the hyperplane A.

Cor.: Through any point outside of a hyperplane there can be drawn one, and only one, line perpendicular to the hyperplane.

50. Shortest Distance of a Point from a Hyperplane:

If D denotes the shortest distance PQ, which is the perpendicular drawn from P to the hyperplane, we have

$$D^2=(x-a)^2+(y-b)^2+(z-c)+(w-d)^2.$$

But

$$\frac{x-a}{a_1}=\frac{y-b}{b_1}=\frac{z-c}{c_1}=\frac{w-d}{d_1}$$

$$=\pm\frac{D}{\sqrt{(a_1{}^2+b_1{}^2+c_1{}^2+d_1{}^2)}}$$

$$=\frac{a_1(x-a)+b_1(y-b)+c_1(z-c)+d_1(w-d)}{(a_1{}^2+b_1{}^2+c_1{}^2+d_1{}^2)}$$

$$\therefore\quad D=\pm\frac{(a_1x+b_1y+c_1z+d_1w)-(aa_1+bb_1+cc_1+dd_1)}{\sqrt{(a_1{}^2+b_1{}^2+c_1{}^2+d_1{}^2)}}$$

$$=\mp\frac{e_1+(aa_1+bb_1+cc_1+dd_1)}{\sqrt{(a_1{}^2+b_1{}^2+c_1{}^2+d_1{}^2)}}. \tag{50.1}$$

51. Projection of a Line on a Hyperplane:

If the points of the line are projected on the hyperplane, the projecting lines are all parallel to the normal direction of the hyperplane, and consequently, they lie in the same plane which intersects the hyperplane in a line. This line of section is the projection of the given line upon the hyperplane.

Let the hyperplane be defined by

$$A \equiv a_1x + b_1y + c_1z + d_1w + e_1 = 0 \qquad \ldots \quad (1)$$

and the given line by

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} = \frac{w-d}{p} = r. \qquad \ldots \quad (2)$$

Let P be any point $(\alpha, \beta, \gamma, \delta)$ on the line (2). If now R be the length of the perpendicular PQ, drawn from P on the hyperplane, then the co-ordinates of the foot Q of this perpendicular are—

$$\alpha + a_1\text{R}, \quad \beta + b_1\text{R}, \quad \gamma + c_1\text{R}, \quad \delta + d_1\text{R}.$$

These must satisfy the equation (1), and we must have

$$a_1(\alpha + a_1\text{R}) + b_1(\beta + b_1\text{R}) + c_1(\gamma + c_1\text{R}) + d_1(\delta + d_1\text{R}) + e_1 = 0$$

or,

$$\text{R}(a_1^2 + b_1^2 + c_1^2 + d_1^2) + (a_1\alpha + b_1\beta + c_1\gamma + d_1\delta + e_1) = 0. \quad (3)$$

But $\quad \alpha = a + lr, \quad \beta = b + mr, \quad \gamma = c + nr, \quad \delta = d + pr$

$$\therefore \quad \text{R}(a_1^2 + b_1^2 + c_1^2 + d_1^2) + \{a_1(a + lr) + b_1(b + mr) + c_1(c + nr) + d_1(d + pr) + e_1\} = 0$$

i.e., $$\text{R}(a_1^2 + b_1^2 + c_1^2 + d_1^2) + r(a_1l + b_1m + c_1n + d_1p) + (aa_1 + bb_1 + cc_1 + dd_1 + e_1) = 0$$

or,

$$\text{R}.\Sigma a_1^2 + r.\Sigma a_1l + \Sigma aa_1 + e_1 = 0,$$

whence

$$\text{R} = -\frac{r\Sigma a_1l + \Sigma aa_1 + e_1}{\Sigma a_1^2} \qquad (51.1$$

and this is the length of the perpendicular drawn from the point P at a distance r from (a, b, c, d) on the line.

If a_1, b_1, c_1, d_1 are the actual direction-cosines, $\Sigma a_1{}^2=1$, and we have

$$R=-(r\Sigma a_1 l+\Sigma aa_1+e_1). \tag{51.2}$$

The co-ordinates of Q are, then,—

$$x=a+lr+a_1R, \quad y=b+mr+b_1R,$$

$$z=c+nr+c_1R, \quad w=d+pr+d_1R.$$

∴ The locus of Q is obtained by eliminating the parameters r and R in the form

$$\left\| \begin{matrix} x-a & y-b & z-c & w-d \\ l & m & n & p \\ a_1 & b_1 & c_1 & d_1 \end{matrix} \right\|=0. \tag{51.3}$$

These are the equations of the projecting plane, *i.e.*, the plane on which the lines projecting the different points of the given line lie. The forms of the equations show that this plane is determined by the given line and the normal to the hyperplane drawn through the point (a, b, c, d). This plane intersects the hyperplane in a line, which is the projection of the given line upon the hyperplane. When, however, the line is perpendicular to the hyperplane, the equation (51.3) is evanescent and there is no projecting plane. In this case the projection is a point—the point where the line intersects the hyperplane.

The length of the perpendicular drawn from (a, b, c, d) on the hyperplane is then given by—

$$R'=-(\Sigma aa_1+e_1) \text{ (50.1)}$$

and the co-ordinates of the foot Q′ of this perpendicular are given by

$$\xi = a + a_1 R' = a - a_1(\Sigma a a_1 + e_1)$$

$$\eta = b + b_1 R' = b - b_1(\Sigma a a_1 + e_1)$$

$$\zeta = c + c_1 R' = c - c_1(\Sigma a a_1 + e_1)$$

$$\tau = d + d_1 R' = d - d_1(\Sigma a a_1 + e_1)$$

The co-ordinates of Q are given by

$$x = a + lr + a_1 R = a + r(l - a_1 \Sigma a_1 l) - a_1(\Sigma a a_1 + e_1)$$

$$= \xi + r(l - a_1 \Sigma a_1 l).$$

Similarly,

$$y = b + r(m - b_1 \Sigma a_1 l) - b_1(\Sigma a a_1 + e_1) = \eta + r(m - b_1 \Sigma a_1 l)$$

$$z = c + r(n - c_1 \Sigma a_1 l) - c_1(\Sigma a a_1 + e_1) = \zeta + r(n - c_1 \Sigma a_1 l)$$

$$w = c + r(p - d_1 \Sigma a_1 l) - d_1(\Sigma a a_1 + e_1) = \tau + r(p - d_1 \Sigma a_1 l)$$

whence the equations of the projection are obtained in the form—

$$\frac{x-\xi}{l - a_1 \Sigma a_1 l} = \frac{y-\eta}{m - b_1 \Sigma a_1 l} = \frac{z-\zeta}{n - c_1 \Sigma a_1 l} = \frac{w-\tau}{p - d_1 \Sigma a_1 l} \quad (51.4$$

showing that the direction-cosines of the projection are proportional to

$$l - a_1 \Sigma a_1 l, \quad m - b_1 \Sigma a_1 l, \quad n - c_1 \Sigma a_1 l, \quad p - d_1 \Sigma a_1 l.$$

Since

$$\Sigma(l - a_1 \Sigma a_1 l)^2 = \Sigma l^2 + (\Sigma a_1 l)^2 \Sigma a_1{}^2 - 2(\Sigma a_1 l)^2$$

$$= 1 + (\Sigma a_1 l)^2 - 2(\Sigma a_1 l)^2$$

$$= 1 - (\Sigma a_1 l)^2$$

$$= \sin^2\phi,$$

where ϕ is the angle between the given line and the normal to the hyperplane, the actual direction-cosines of the projection are

$$\frac{l-a_1\Sigma a_1 l}{\sin\phi},\ \frac{m-b_1\Sigma a_1 l}{\sin\phi},\ \frac{n-c_1\Sigma a_1 l}{\sin\phi},\ \frac{p-d_1\Sigma a_1 l}{\sin\phi}. \quad (51.5$$

52. Projection of a Plane upon a Hyperplane:

The projecting lines of the points of the plane are all in the normal direction of the hyperplane and are, therefore, parallel and lie in one and the same hyperplane. This hyperplane meets the given hyperplane in a plane, which may be called the *projection* of the plane upon the hyperplane.

Let the given hyperplane be defined by

$$A \equiv \Sigma lx - k = 0 \quad \dots \quad (1)$$

and the given plane by

$$B \equiv \Sigma l'x - k' = 0 \quad \dots \quad (2) \quad \text{and} \quad C \equiv \Sigma l''x - k'' = 0. \quad \dots \quad (3)$$

Any hyperplane containing the given plane is then represented by

$$B + \lambda C = 0, \text{ where } \lambda \text{ is a parameter.} \quad \dots \quad (4)$$

If the hyperplane (4) is to be the projecting hyperplane, it must contain the normal direction of the given hyperplane (1). This requires that

$$\Sigma l(l' + \lambda l'') = 0, \quad i.e., \quad \Sigma ll' + \lambda \Sigma ll'' = 0. \quad \dots \quad (5)$$

Eliminating λ between (4) and (5), the equation of the projecting hyperplane is obtained in the form

$$B\Sigma ll'' - C\Sigma ll' = 0. \quad (52.1$$

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Hence, the plane of projection is given by the equations (1) and (52.1).

If the plane be defined in the form—

$$\left\| \begin{matrix} x-a & y-b & z-c & w-d \\ l' & m' & n' & p' \\ l'' & m'' & n'' & p'' \end{matrix} \right\| = 0 \qquad \dots \quad (6)$$

the projecting hyperplane contains the plane, *i.e.*, contains the lines (l, m, n, p) and (l', m', n', p') through (a, b, c, d), and also the normal direction of the given hyperplane. Hence, its equation takes the form (34.3)—

$$\left| \begin{matrix} x-a & y-b & z-c & w-d \\ l' & m' & n' & p' \\ l'' & m'' & n'' & p'' \\ l & m & n & p \end{matrix} \right| = 0. \qquad (52.2$$

Equations (1) and (52.2) determine the plane of projection, and since the given plane lies with its projection in the same hyperplane (52.2), they intersect in a line, which also lies in the given hyperplane. In fact, it is the line in which the given plane (6) intersects the hyperplane (1).

53. Projection of a Point on a Plane:

Let the plane be defined by the two equations

$$\left. \begin{aligned} A \equiv \Sigma lx - p = 0 \\ B \equiv \Sigma mx - q = 0 \end{aligned} \right\} \quad \dots \qquad \dots \quad (1)$$

Let P(a, b, c, d) be the given point and Q(x, y, z, w) be the foot of the perpendicular drawn from P on the plane.

By considering a point Q′ in the plane, very near to the point Q, we obtain

$$\Sigma l\delta x=0 \quad \ldots \quad (2) \qquad \text{and} \qquad \Sigma m\delta x=0. \quad \ldots \quad (3)$$

Also, $\quad PQ^2=(x-a)^2+(y-b)^2+(z-c)^2+(w-d)^2$

whence, $(x-a)\delta x+(y-b)\delta y+(z-c)\delta z+(w-d)\delta w=0 \ldots \quad (4)$

From relations (2), (3) and (4), we obtain

$$\begin{aligned} x-a&=\lambda l_1+\mu m_1 \qquad & z-c&=\lambda l_3+\mu m_3 \\ y-b&=\lambda l_2+\mu m_2 \qquad & w-d&=\lambda l_4+\mu m_4 \qquad \ldots \quad (5) \end{aligned}$$

where λ and μ are two parameters.

Eliminating λ and μ between these equations, we obtain—

$$\left\| \begin{matrix} x-a & y-b & z-c & w-d \\ l_1 & l_2 & l_3 & l_4 \\ m_1 & m_2 & m_3 & m_4 \end{matrix} \right\| =0. \quad \ldots \quad (6)$$

Equations (6) represent a plane containing the two normal directions of the hyperplanes, which define the plane, and this plane intersects the given plane in a point Q, which is the required projection.

The co-ordinates of Q are obtained from the four equations (1) and (6).

If, however, Q(x, y, z, w) is assumed to be a fixed point and P(a, b, c, d) a variable point, the equations (6) represent the locus of all points whose projections on the given plane coincide with Q. But these two

equations (6) represent a plane, and it follows, therefore, that the locus of all points whose projections on a plane coincide with a fixed point Q is a plane, or in other words, the lines joining points of the plane (6) with Q are all perpendicular to the given plane.

Hence, in the fourfold we meet with another strange fact that, at any point of a given plane there is an infinitude of normals to the plane, all lying in another plane.

The planes (1) and (6) are endowed with very interesting properties peculiar to the fourfold, the analogue not being found in the ordinary space.

They are so related that all lines in one are perpendicular to all lines in the other. For, let $(\lambda, \mu, \nu, \rho)$ be a line in (1) and $(\lambda', \mu', \nu', \rho')$ a line in (6).

Then, we may take

$$\lambda' = \alpha l_1 + \beta m_1 \qquad \nu' = \alpha l_3 + \beta m_3$$

$$\mu' = \alpha l_2 + \beta m_2 \qquad \rho' = \alpha l_4 + \beta m_4.$$

Since $(\lambda, \mu, \nu, \rho)$ lies in (1), we must have—

$$\Sigma\lambda l = 0 \qquad \text{and} \qquad \Sigma\lambda m = 0$$

Now,
$$\Sigma\lambda\lambda' = \lambda(\alpha l_1 + \beta m_1) + \mu(\alpha l_2 + \beta m_2) + \nu(\alpha l_3 + \beta m_3) + \rho(\alpha l_4 + \beta m_4)$$

$$= \alpha\Sigma\lambda l + \beta\Sigma\lambda m = 0.$$

It is to be noted then that, from an external point P, only one perpendicular can be drawn to a plane, but at the foot of this perpendicular an infinite number of lines can be drawn, all normal to the plane and lying in another plane.

These two planes are called "absolutely" or "completely" perpendicular planes, and it follows that through an external point, only one absolutely perpendicular plane can be drawn to any plane.

Cor.: At every point of a plane, there is more than one line perpendicular to it, all contained in the absolutely perpendicular plane, and it possesses properties similar to those of a normal to a plane in the ordinary space.

Note.—It is clear now that the normals to the two hyperplanes defining a plane at a common point are perpendicular to the plane and determine the absolutely perpendicular plane to the given plane at that point. It is to be noted that two absolutely perpendicular planes have only one point in common and never intersect in a line; we can never see them both simultaneously. The most that can be seen is one of the planes and a single line of the other. At different points of a plane, however, there are different absolutely perpendicular planes.

54. The shortest Distance of a Point from a Plane:

The length of the perpendicular PQ drawn from $P(a, b, c, d)$ to the plane is the shortest distance between the point and the plane.

If, then, the co-ordinates (ξ, η, ζ, ρ) of Q are determined by means of equations (1) and (6) of §53, the shortest distance D is given by

$$D=(\xi-a)^2+(\eta-b)^2+(\zeta-c)^2+(\rho-d)^2 \qquad (54.1$$

If (l_1, m_1, n_1, p_1) and (l_2, m_2, n_2, p_2) are the guiding lines of the plane through the fixed point

$O(\alpha, \beta, \gamma\delta,)$, including an angle ω, and D_1, D_2 are the projections of OP on the guiding lines,

$$D^2 = \Sigma(a-\alpha)^2 - \frac{1}{\sin^2\omega}(D_1^2 - 2D_1D_2\cos\omega + D_2^2).* \qquad (54.2$$

55. Projection of a Line upon a Plane:

Here, the projecting lines of the different points of the line are not parallel to one and the same direction, as the given plane has no unique normal direction. All these projecting lines, therefore, are contained in a hyperplane, which intersects the given plane in a line, and this is the required projection of the given line upon the given plane.

Let the given line be defined by

$$\frac{x-a}{L} = \frac{y-b}{M} = \frac{z-c}{N} = \frac{w-d}{P} \qquad \dots \quad (1)$$

and the plane on which it is to be projected be defined by

$$\Sigma lx - k = 0, \qquad \Sigma mx - k' = 0. \qquad \dots \quad (2)$$

The direction-cosines of the projecting line of any point $P(\alpha, \beta, \gamma, \delta)$ on (1) may be taken as

$$\lambda l_1 + \mu m_1, \quad \lambda l_2 + \mu m_2, \quad \lambda l_3 + \mu m_3, \quad \lambda l_4 + \mu m_4,$$

since this line is parallel to a direction in the plane of the normals to the two hyperplanes (2).

Hence, if $Ax + By + Cz + Dw + E = 0 \qquad \dots \quad (3)$

be the equation of the hyperplane determined by the

* A. R. Forsyth, Geometry of Four Dimensions, Vol. I, p. 50.

projecting lines and the given line, we must have

$$AL + BM + CN + DP = 0$$

$$Al_1 + Bl_2 + Cl_3 + Dl_4 = 0$$

$$Am_1 + Bm_2 + Cm_3 + Dm_4 = 0$$

Also,

$$Aa + Bb + Cc + Dd + E = 0.$$

Eliminating A, B, C, D, E between these equations, we obtain the equation of the projecting hyperplane in the form

$$\begin{vmatrix} x-a & y-b & z-c & w-d \\ L & M & N & P \\ l_1 & l_2 & l_3 & l_4 \\ m_1 & m_2 & m_3 & m_4 \end{vmatrix} = 0. \qquad (55.1$$

The equations (2) and (55.1) will determine the line of projection.

It is to be noted that the locus of the projecting lines of the different points of the line is the hyperplane (55.1). Since they all intersect two non-intersecting lines, they are generators of a hyperboloid of one sheet.*

Note.—If the line lies in an absolutely perpendicular plane to the given plane, the projecting lines all coincide with the line, and the projection, therefore, coincides with the point in which the given line meets the plane.

56. Theorem: *Two lines are given in general position. Show that a plane can be passed through either upon which it will be the projection of the other.*

* Salmon, *loc. cit.*, § 109.

Let $$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}=\frac{w-d}{p}$$

and $$\frac{x-a'}{l'}=\frac{y-b}{m'}=\frac{z-c'}{n'}=\frac{w-d'}{p'}$$

be the two given lines.

The equation of the hyperplane determined by these lines is

$$\begin{vmatrix} x-a & y-b & z-c & w-d \\ a-a' & b-b' & c-c' & d-d' \\ l & m & n & p \\ l' & m' & n' & p' \end{vmatrix}=0$$

or, $$L(x-a)+M(y-b)+N(z-c)+P(w-d)=0. \quad \ldots \quad (1)$$

Now, at (a', b', c', d') of the second line, draw the normal to this hyperplane. The equations of the plane determined by the second line and this normal are then

$$\left\Vert \begin{matrix} x-a' & y-b' & z-c' & w-d' \\ l' & m' & n' & p' \\ L & M & N & P \end{matrix} \right\Vert=0. \quad (56.1$$

This is the required plane ; for, this plane is perpendicular to the hyperplane * (1), and the lines projecting the points of the first line are, therefore, perpendicular to the plane (56.1), and intersect this plane along the given second line. Hence, the second line is the projection of the first line on the plane (56.1).

* Since the plane contains the normal direction of the hyperplane. See § 69.

57. Projection of a Point on a Line :

Let $A \equiv \Sigma lx - p = 0, \qquad B \equiv \Sigma mx - q = 0$

$$C \equiv \Sigma nx - r = 0 \qquad \ldots \quad (1)$$

denote the given line.

The projection Q of a point $P(a, b, c, d)$ on this line is obtained by drawing the perpendicular PQ on the line.

Proceeding exactly as in § 53, we find

$$PQ^2 = (x-a)^2 + (y-b)^2 + (z-c)^2 + (w-d)^2 \qquad \ldots \quad (2)$$

and

$$\left.\begin{aligned} x-a &= \alpha l_1 + \beta m_1 + \gamma n_1 \\ y-b &= \alpha l_2 + \beta m_2 + \gamma n_2 \\ z-c &= \alpha l_3 + \beta m_3 + \gamma n_3 \\ w-d &= \alpha l_4 + \beta m_4 + \gamma n_4 \end{aligned}\right\} \qquad \ldots \quad (3)$$

Eliminating α, β, γ between the equations (3), the equation of a hyperplane, which intersects the given line in the required point Q, is obtained in the form

$$\begin{vmatrix} x-a & y-b & z-c & w-d \\ l_1 & l_2 & l_3 & l_4 \\ m_1 & m_2 & m_3 & m_4 \\ n_1 & n_2 & n_3 & n_4 \end{vmatrix} = 0 \qquad (57.1$$

showing that this hyperplane passes through the point P and its guiding lines are parallel to the three normals to the three given hyperplanes defining the line, and consequently, this hyperplane is orthogonal to the given line and meets it in the required point Q.

It follows also that the locus of points, whose projections on a line coincide with one point, is a hyperplane at right angles to the line, or, in other words, *all lines at right angles to a given line at a given point of it is a hyperplane orthogonal to the line.*

If the line be defined by the equations

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=\frac{w-\delta}{p}=r \qquad \dots \quad (4)$$

and the point be $P(a, b, c, d)$, then the co-ordinates of any point R on the line may be taken as

$$\alpha+lr, \quad \beta+mr, \quad \gamma+nr, \quad \delta+pr$$

$$PR^2=(a-\alpha-lr)^2+(b-\beta-mr)^2+(c-\gamma-nr)^2+(d-\delta-pr)^2$$

$$=\Sigma(a-\alpha)^2-2r\Sigma l(a-\alpha)+r^2 \qquad \dots \qquad \dots \quad (5)$$

Now, PR will be a minimum, when PR is perpendicular to the line.

Differentiating with respect to r, we get

$$-\Sigma l(a-\alpha)+r=0, \qquad or, \quad r=\Sigma l(a-\alpha)$$

$\therefore$ For this value of r, we have, then,

$$PR^2=\Sigma(a-\alpha)^2-2\{\Sigma l(a-\alpha)\}^2+\{\Sigma l(a-\alpha)\}^2$$

$$=\Sigma(a-\alpha)^2-\{\Sigma l(a-\alpha)\}^2$$

$\therefore$ If D denotes the perpendicular, we have

$$D=[\Sigma(a-\alpha)^2-\{\Sigma l(a-\alpha)\}^2]^{\frac{1}{2}}. \qquad (57.2$$

58. The shortest Distance between two Lines:

Let the two lines be defined by

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}=\frac{w-d}{p}=r \qquad \dots \quad (1)$$

$$\frac{x-a'}{l'}=\frac{y-b'}{m'}=\frac{z-c'}{n'}=\frac{w-d'}{p'}=r'. \qquad \dots \quad (2)$$

The shortest distance lies on a line which intersects both and is perpendicular to both.

Let $(\lambda, \mu, \nu, \rho)$ be the direction-cosines of the common perpendicular line of the two given lines and suppose it meets them in A and B respectively.

Let P and Q be any two points taken respectively on the two given lines, so that P is the point

$$(a+lr, \quad b+mr, \quad c+nr, \quad d+pr)$$

and Q is the point

$$(a'+l'r', \quad b'+m'r', \quad c'+n'r', \quad d'+p'r')$$

$$\therefore \qquad PQ^2=\Sigma(a'-a+l'r'-lr)^2. \qquad \dots \quad (3)$$

If, then, PQ is to be the shortest distance, its differential co-efficients with respect to r, r' must vanish, so that the critical equations thus obtained give

$$-\Sigma l(a'-a+l'r'-lr)=0 \qquad \dots \quad (4)$$

$$\Sigma l'(a'-a+l'r'-lr)=0. \qquad \dots \quad (5)$$

Equations (4) and (5) will supply values of r and r' for which PQ will be a minimum. Hence, if D denotes the shortest distance, we have

$$D^2=\Sigma(a'-a+l'r'-lr)^2 \qquad (58.1$$

where r and r' are given by the relations (4) and (5).

Note.—If $(\lambda, \mu, \nu, \rho)$ are the direction-cosines of the shortest distance, we obtain

$$\Sigma l\lambda = 0, \qquad \Sigma l'\lambda = 0$$

but these two equations are not sufficient to determine $(\lambda, \mu, \nu, \rho)$, the reason being that in the fourfold, there is no unique direction which is perpendicular to two given directions ; in fact, there is an infinite number, lying in a plane, as already pointed out.

Alternative Method :

We may proceed to determine the shortest distance by another method :

The two lines lie in a hyperplane and the problem may be reduced to one in a hyperplane.

Let the given points be denoted by $A(a, b, c, d)$ and $B(a', b', c', d')$. Take two points P and Q on the lines at distances r and r' from A and B respectively. Then ABPQ is a tetrahedron.

If V denotes the volume of this tetrahedron, we have

$$6V = rr'\delta \sin\theta \qquad \ldots \quad (6)$$

where r, r' are two opposite edges, δ the shortest distance between them, and θ the angle between the two edges.

The content of the finite join of ABPQ is given by

$$(6V)^{2*} = \Sigma \begin{vmatrix} a+lr & b+mr & c+nr & 1 \\ a'+l'r' & b'+m'r' & c'+n'r' & 1 \\ a & b & c & 1 \\ a' & b' & c' & 1 \end{vmatrix}^2$$

* The square of the volume of the tetrahedron is equal to the sum of the squares of its projections on the four co-ordinate hyperplanes.

$$=r^2r'^2\Sigma\begin{vmatrix} a-a' & b-b' & c-c' \\ l & m & n \\ l' & m' & n' \end{vmatrix}^2 \quad \dots \quad (7)$$

From (6) and (7), we find

$$\delta^2\sin^2\theta=\Sigma\begin{vmatrix} a-a' & b-b' & c-c' \\ l & m & n \\ l' & m' & n' \end{vmatrix}^2 \quad (58.2$$

Note.—It is to be noted that in the Euclidean Geometry, only one common perpendicular line exists between any two lines not lying in the same plane. In the fourfold, although there are more than one common perpendicular directions of two such lines, there is only one which intersects them both and is perpendicular to both. In the non-Euclidean Elliptic Geometry, however, there may be infinite number of common perpendiculars to two non-coplanar lines, on which they cut out equal segments.

59. The shortest Distance between a Line and a Plane :

When a line and a plane do not lie in the same hyperplane, they do not intersect, and consequently, there is a point in the line whose distance from the plane is less than or equal to the distance of any other point from the plane, and the line of this minimum distance is perpendicular to both.

Suppose the line is defined by the equations

$$\frac{x-a}{\lambda}=\frac{y-b}{\mu}=\frac{z-c}{\nu}=\frac{w-d}{\rho}=r \quad \dots \quad (1)$$

and the plane is defined by

$$\left\| \begin{matrix} x-\alpha & y-\beta & z-\gamma & w-\delta \\ l & m & n & p \\ l' & m' & n' & p' \end{matrix} \right\| = 0 \quad \ldots \quad (2)$$

Take any point $P(a+\lambda r,\ b+\mu r,\ c+\nu r,\ d+\rho r)$ on the line (1) and any point $Q(\alpha+kl+k'l',\ \beta+km+k'm',\ \gamma+kn+k'n',\ \delta+kp+k'p')$ on (2). (See 32.5.)

Then

$$PQ^2 = \Sigma(a+\lambda r-\alpha-kl-k'l')^2. \quad \ldots \quad (3)$$

If, then, PQ is to be a minimum, the critical equations are

$$\left.\begin{aligned} \Sigma(a-\alpha+\lambda r-kl-k'l')\lambda &= 0 \\ -\Sigma(a-\alpha+\lambda r-kl-k'l')l &= 0 \\ -\Sigma(a-\alpha+\lambda r-kl-k'l')l' &= 0 \end{aligned}\right\} \quad \ldots \quad (4)$$

Equations (4) will determine the critical values of the three parameters r, k and k' ; substituting these values in (3), the shortest distance D is obtained in the form

$$D = \{\Sigma(a-\alpha+\lambda r-kl-k'l')^2\}^{\frac{1}{2}}. \quad (59.1$$

We can determine the shortest distance in another manner, namely, by considering this as the projection on the common perpendicular line of the line joining the point $A(a, b, c, d)$ on the given line and the point $B(\alpha, \beta, \gamma, \delta)$ in the plane, so that, if L, M, N, P

be the direction-cosines of the common perpendicular line, we have

$$D=(a-\alpha)L+(b-\beta)M+(c-\gamma)N+(d-\delta)P. \quad \ldots \quad (5)$$

But L, M, N, P are determined from the conditions that the line is perpendicular to the given line, as well as to the two guiding lines in the given plane.

Thus

$$\left.\begin{aligned} \lambda L+\mu M+\nu N+\rho P&=0 \\ Ll+Mm+Nn+Pp&=0 \\ Ll'+Mm'+Nn'+Pp'&=0 \end{aligned}\right\} \quad \ldots \quad (6)$$

Solving equations (6) for L, M, N, P, we get

$$\frac{L}{\begin{vmatrix} \mu & \nu & \rho \\ m & n & p \\ m' & n' & p' \end{vmatrix}} = \frac{M}{-\begin{vmatrix} \lambda & \nu & \rho \\ l & n & p \\ l' & n' & p' \end{vmatrix}}$$

$$=\frac{N}{\begin{vmatrix} \lambda & \mu & \rho \\ l & m & p \\ l' & m' & p' \end{vmatrix}} = \frac{P}{-\begin{vmatrix} \lambda & \mu & \nu \\ l & m & n \\ l' & m' & n' \end{vmatrix}} = \frac{1}{\vartheta} \text{ (say)}$$

where $\vartheta^2=\Sigma\begin{vmatrix} \mu & \nu & \rho \\ m & n & p \\ m' & n' & p' \end{vmatrix}^2 = \begin{vmatrix} \Sigma\lambda^2 & \Sigma l\lambda & \Sigma l'\lambda \\ \Sigma l\lambda & \Sigma l^2 & \Sigma ll' \\ \Sigma l'\lambda & \Sigma ll' & \Sigma l'^2 \end{vmatrix}$

$$= \begin{vmatrix} 1 & \cos\theta & \cos\phi \\ \cos\theta & 1 & \cos\psi \\ \cos\phi & \cos\psi & 1 \end{vmatrix} \equiv \Theta \text{ (say)}$$

where θ, ϕ, ψ are the mutual angles between the three lines.

Substituting these values in (5), we get

$$D = L(a-\alpha) + M(b-\beta) + N(c-\gamma) + P(d-\delta)$$

i.e., $$D = \begin{vmatrix} a-\alpha & b-\beta & c-\gamma & d-\delta \\ \lambda & \mu & \nu & \rho \\ l & m & n & p \\ l' & m' & n' & p' \end{vmatrix} \div \Theta^{\frac{1}{2}} \qquad (59.2$$

If the line intersects the plane, $D=0$ and this is equivalent to the conditions (42.5).

Note.—The plane, determined by the given line and the common perpendicular, meets the given plane at the point where the common perpendicular meets it. This plane is consequently perpendicular to the given plane ; and since it does not meet the given plane in any other point, it will be absolutely perpendicular to the given plane at the point. Any other plane, drawn through the common perpendicular line, will meet the given plane in lines and will be perpendicular to it.

CHAPTER VI

ANGLE-CONCEPT IN THE FOURFOLD

60. Measurement of Hyper-angles :

The metrical study of hyper-angles, a subject analogous to Trigonometry of the ordinary space of three dimensions (the hyperplane of the fourfold) presents considerable difficulties, in view of the fact that the entities cannot be visualised, and consequently, the subject is still in its infancy. Serious attention of geometers, however, is gradually being diverted to logical development of the various intricate formulæ applicable to angles of higher dimensions. The subject is capable of producing very interesting and beautiful results in manifolds of any number of dimensions by a natural and logical extension of the formulæ in lower geometries. The lines, planes and hyperplanes, which are the only linear manifolds in the fourfold, will engage our attention for the present ; and to this end, new varieties of angular measurements will have to be introduced.

61. Angular Invariants :

M. Camille Jordan * has proved for a manifold of n dimensions that there are certain functions of the parameters, used in the defining equations of spaces,

* C. Jordan, Essai sur la géométrie à n dimensions, Bull. de la Soc. Math. de France (1875), Tome II, p. 109.

which remain invariant for any orthogonal transformation of co-ordinates. These functions are defined as "angular invariants" of the spaces, but it must be carefully noted that the trigonometrical angles, as conceived in the lower spaces, do not remain invariant, for their signs may be changed with the transformation applied. Although these functions are not much helpful in producing any geometric conception of the inclination of the geometric linear or other spaces, they suggest a very convenient mode for analytic representation thereof ; and in particular, angular measurements in the fourfold can, with sufficient ease, be expressed in terms of these functions.

62. Conception of the Hyperplane Angle:

It has been pointed out that the locus of the common points of two hyperplanes is a plane, which may be called the "plane of cleavage" between the two hyperplanes. This plane divides each of the hyperplanes into two half-hyperplanes, just as a point divides a line into two half-lines, and a line in a plane divides it into two half-planes. Thus, two half-hyperplanes having a common face may be taken to constitute what may be called a *hyperplane angle*—a notion to be introduced in the fourfold. It is a logical extension of the notion of a 'plane angle' and a 'dihedral angle.' Two lines meeting in a point is said to form a plane angle, and two planes meeting in a line forms a dihedral angle. As a logical extension, two hyperplanes meeting in a plane may be conceived to form an angle of a new and higher denomination, the nature of which will be manifest on a closer examination

of the notion, which is a necessity in the geometry of the fourfold. The two half-hyperplanes are called the 'cells' of the hyperplane angle, and the plane of cleavage may be called its 'face.'

The two hyperplanes have each a determinate normal direction ; consequently at any point of the face, the two normals to the two hyperplanes, which are orthogonal to the common face, will determine a plane, which intersects the hyperplanes in two lines, forming an angle which remains invariant for all positions of the point in the face. This constant angle may be called the *plane angle* of the hyperplane angle, and this can be taken as a measure of the hyperplane angle, just as a dihedral angle is measured by the plane angle formed by the two lines in which the plane of the two normals to the faces at any common point of their edge cuts the two planes. It will be easily recognised that the plane of this plane angle is absolutely perpendicular to the face, and intersects the two constituent hyperplanes in the arms of the plane angle. It follows, therefore, that the absolutely perpendicular plane at different points of the face intersect the two hyperplanes in lines forming a constant angle.

63. Inclination of two Hyperplanes:

By *inclination* of two geometric entities is meant the operation or operations by which one may be made to coincide with or brought to the position of the other. Thus, the inclination of two lines denotes the rotation of one around their common point into the position of the other ; the inclination of two

planes denotes the rotation of one about the common edge into the position of the other. The inclination of a line to a plane is the rotation of the line about the common point by which the line is made to coincide with the plane. By an analogous inference, the inclination of two hyperplanes may be defined as the operation which will bring the two hyperplanes into coincidence. This operation, which is certainly a rotation of one of them around the common face—(a phenomenon peculiar to the fourfold, its rationale will be established subsequently) which brings the two hyperplanes into coincidence, will at the same time bring their two normals into coincidence. Thus, it appears that the angle between the two normals may be taken as a measure of the operation needed to effect the coincidence of the two constituent hyperplanes. The inclination between the two normals to the two hyperplanes at any common point may consequently be taken as a measure of the inclination of the two hyperplanes.

From what has been stated above, we are now in a position to postulate that the inclination of two hyperplanes is measured by the hyperplane angle between them and this again is measured by its plane angle, *i.e.*, by the angle between the two normals to the cells.

$$\text{Let} \quad A \equiv lx + my + nz + pw + k = 0$$

$$\text{and} \quad B \equiv l'x + m'y + n'z + p'w + k' = 0$$

be the defining equations of two hyperplanes. Denoting the angle between their two normals at any common point by θ, we have

$$\text{Cos}\ \theta = ll' + mm' + nn' + pp'. \qquad (63.1$$

The angle θ may be determined also in another manner. The plane of the two normals at any point $O(a, b, c, d)$ of the face is given by the equations

$$\left\| \begin{matrix} x-a & y-b & z-c & w-d \\ l & m & n & p \\ l' & m' & n' & p' \end{matrix} \right\| = 0. \quad (63.2$$

This plane meets the two given hyperplanes in two lines, which are coplanar with and respectively perpendicular to the two normals, and consequently, they include the same angle between them as do the two normals. In fact, the angle θ is measured by the angle between the two lines in which the plane of the two normals intersects the half-hyperplanes ; and it will be seen later that the plane intersects both the hyperplanes orthogonally and is the plane absolutely perpendicular to the face at O.

Take a direction (l_1, m_1, n_1, p_1) in the first hyperplane and another (l_2, m_2, n_2, p_2) in the second. If ϕ denotes the angle between them, we have

$$\cos \phi = l_1 l_2 + m_1 m_2 + n_1 n_2 + p_1 p_2,$$

$$\Sigma l_1 l = 0, \quad \Sigma l_1{}^2 = 1; \quad \Sigma l_2 l' = 0, \quad \Sigma l_2{}^2 = 1.$$

The value of $\cos^2 \phi$ can be made zero, when the two lines coincide in the common face but it can never exceed 1. Hence, $\cos^2 \phi$ always has a maximum value between 0 and 1. It can be shown that $\cos^2 \phi$ attains this maximum value, when the directions (l_1, m_1, n_1, p_1) and (l_2, m_2, n_2, p_2) lie in the plane of the two normals, and then ϕ attains the value θ, which is the angle between the normals. In fact, the two normals and

the two directions are coplanar in the plane (63.2), which is orthogonal to both the hyperplanes.

Thus, θ measures the hyperplane angle between the two hyperplanes, which is the inclination required.

Cor. 1: The two hyperplanes will be coincident (or parallel) when $\theta=0$, *i.e.*, when their normal directions are parallel ;

for, $\quad \Sigma ll' = ll' + mm' + nn' + pp' = \cos 0 = 1$

whence $\quad \sin \theta = \Sigma (lm' - l'm)^2 = 0.$

Since the quantities l's and l''s are all real, the required conditions are

$$\frac{l}{l'} = \frac{m}{m'} = \frac{n}{n'} = \frac{p}{p'} \tag{63.3}$$

i.e., the hyperplanes have the same normal direction.

Cor. 2: The two hyperplanes will be mutually orthogonal, if $\theta=\pi/2$, *i.e.*, if $\Sigma ll'=0$, showing that their normals are also orthogonal, *i.e.*, each hyperplane contains a normal direction of the other.

The hyperplane angle between two orthogonal hyperplanes may, therefore, be termed a *right hyperplane angle* and is measured by a right plane angle.

64. Angular Invariant of two Hyperplanes :

We have already determined the expression for the angle between the two hyperplanes in the form

$$\cos \theta = \Sigma ll' \text{ and } \sin^2 \theta = 1 - \cos^2 \theta$$
$$= 1 - (\Sigma ll')^2 = \Sigma l^2 . \Sigma l'^2 - (\Sigma ll')^2$$

$$\therefore \quad \tan^2 \theta = \frac{\Sigma l^2 . \Sigma l'^2 - (\Sigma ll')^2}{(\Sigma ll')^2} . \tag{64.1}$$

The function on the right-hand side is easily found to be an invariant for an orthogonal transformation and is called an *angular invariant* for the two hyperplanes.

By a proper choice of co-ordinates, the equations of the two hyperplanes may be put in simpler forms. We may take, for instance, the hyperplane $x=0$ (*i.e.*, that determined by the three remaining axes of y, z and w) for the first hyperplane, so that $A \equiv x=0$.

If now the plane of cleavage of the two given hyperplanes be taken for the co-ordinate plane of zw (*i.e.*, $x=0$, $y=0$), then the second hyperplane may be represented by the equation

$$B \equiv x + \lambda y = 0, \text{ where } \lambda \text{ is a parameter.}$$

Putting, now, $\lambda = -\cot \phi$, the equation of the two hyperplanes may be reduced to the forms

$$A \equiv x=0, \quad B \equiv y - x \tan \phi = 0.$$

If from any point P on the first, a perpendicular PQ be drawn on the other (so that $PQ=y$), and at Q, the foot of this perpendicular in the common face, a second perpendicular $QO=x$ be drawn to the face, then

$$\frac{PQ}{QO} = \frac{y}{x} = \tan \phi$$

i.e., the ratio of these perpendiculars is the tangent of the angle between the normals to the two hyperplanes, and ϕ, therefore, may be taken as a measure of the angle between them.

The above formula now becomes

$$\tan^2 \theta = \frac{1-\sin^2 \phi}{\sin^2 \phi} = \operatorname{cosec}^2 \phi - 1 = \cot^2 \phi$$

$$= \tan^2 \left(\frac{\pi}{2} - \phi\right). \tag{64.2}$$

Hence, if A be a half-hyperplane containing a plane α, a second half-hyperplane B can be drawn through α making an angle ϕ with A. At any point O of α, draw the absolutely perpendicular plane cutting A in a line OM. At any point M of this line draw MQ perpendicular to A, such that $MQ = OM. \tan \phi$. Then the hyperplane determined by α and Q is the required hyperplane B, making an angle ϕ with A.

65. Theorem: *The plane angle of a hyperplane angle is constant for all points of the face.*

From what has been stated in §§ 62 and 63, it is now easily seen that the arms of the plane angle are the lines in which the absolutely perpendicular plane at any point of the face intersects the two cells, and they are respectively perpendicular to the two fixed normal directions lying in their plane. Hence, they include an angle equal to that between the fixed normal directions, and this angle is, therefore, invariant for all positions of the point.

Since $\cos \theta = ll' + mm' + nn' + pp'$, the value of θ remains invariant, so long l, m, n, p and l', m', n', p' are constants.

Note.—Owing to the invariant nature of the plane angle of a hyperplane angle, the latter may always be regarded as a magnitude and compared with others of the same denomination. Thus, we may divide a hyperplane angle into parts, etc.

66. Theorem: *Two hyperplane angles are congruent, if the plane angle of one is equal to the plane angle of the other.*

Since each hyperplane angle has a constant plane angle for its measure, it follows that two hyperplane angles may be said to be equal, when their plane angles are equal. This may be taken as the definition of the equality of two hyperplane angles, so far as their magnitudes are concerned. Geometrically, the cells of each are normal to the arms of the plane angle. When, therefore, the plane angles of two hyperplane angles are equal, their arms may be made to coincide, and consequently, the cells of the hyperplane angles will also coincide, each to each. In fact, the half-hyperplanes of one will coincide respectively with those of the other, and we say that the two hyperplane angles are congruent.

Ex. 1. Prove that when two hyperplanes are perpendicular to one another, each contains a normal direction of the other.

Ex. 2. If a hyperplane is perpendicular to two other hyperplanes, its normal lies in the plane of cleavage of these latter.

Ex. 3. The locus of points equidistant from the *cells* of a hyperplane angle is a half-hyperplane through the *face*, bisecting the hyperplane angle.

67. Inclination of a Line to a Hyperplane:

The inclination of a line to a hyperplane may be taken as the least angle which the line makes with any line lying in the hyperplane, or the angle between the line and its projection on the hyperplane, or otherwise, the inclination of a line to a hyperplane may be measured by the angle which the line makes with the

normal direction of the hyperplane ; for, this angle is the complement of the angle which the line makes with its projection.

Let

$$lx+my+nz+pw=k \qquad \dots \quad (1)$$

define the given hyperplane and let the given line be defined by

$$\frac{x-a}{l'}=\frac{y-b}{m'}=\frac{z-c}{n'}=\frac{w-d}{p'}=r. \qquad \dots \quad (2)$$

Suppose the line intersects the hyperplane (1) in the point A(a, b, c, d). Through A draw a direction $(\lambda, \mu, \nu, \rho)$ in the hyperplane. If ψ denotes the angle between the given line and this direction, we have

$$\cos\psi=\lambda l'+\mu m'+\nu n'+\rho p' \qquad \dots \quad (3)$$

subject to the conditions

$$\Sigma\,\lambda^2=1 \quad \dots \quad (4) \qquad \text{and} \qquad \Sigma\,\lambda\,l=0. \quad \dots \quad (5)$$

In order to avoid ambiguity of sign, which is dependent on the sense in which the direction is specified, we consider the magnitude $\cos^2\psi$, which can never be negative, though it can be zero, and it can never exceed unity, *i.e.*, $\cos^2\psi$ must lie between 0 and 1.

In order to determine the minimum value of ψ, the critical equations are obtained by differentiating equations (3), (4) and (5) in the forms—

$$\left.\begin{aligned} l'\delta\lambda+m'\delta\mu+n'\delta\nu+p'\delta\rho&=0\\ \lambda\delta\lambda+\mu\delta\mu+\nu\delta\nu+\rho\delta\rho&=0\\ l\delta\lambda+m\delta\mu+n\delta\nu+p\delta\rho&=0 \end{aligned}\right\}$$

whence we immediately get

$$\lambda = \alpha l + \beta l', \qquad \mu = \alpha m + \beta m',$$

$$\nu = \alpha n + \beta n', \qquad \rho = \alpha p + \beta p'$$

where α, β are any two parameters.

These show that the critical direction lies in the plane determined by the given line and the normal to the hyperplane.

From what has been said in § 51, it follows then that the critical line is the line of intersection of this plane with the hyperplane and is, consequently, the projection of the given line on the hyperplane.

The direction-cosines of the projection are given by (51.5)

$$\frac{l' - l\Sigma ll'}{\sin\phi}, \quad \frac{m' - m\Sigma ll'}{\sin\phi}, \quad \frac{n' - n\Sigma ll'}{\sin\phi}, \quad \frac{p' - p\Sigma ll'}{\sin\phi},$$

where ϕ is the angle between the given line and the normal to the hyperplane.

Hence, the angle θ between the given line and its projection is given by

$$\cos\theta = \Sigma l' . \frac{(l' - l\Sigma ll')}{\sin\phi} = \frac{\Sigma l'^2 - (\Sigma ll')^2}{\sin\phi}$$

$$= \frac{1 - \cos^2\phi}{\sin\phi} = \frac{\sin^2\phi}{\sin\phi} = \sin\phi, \qquad (67.1$$

showing that the required angle is the complement of the angle which the line makes with the normal.

If the hyperplane be defined by three concurrent lines (l_1, m_1, n_1, p_1), (l_2, m_2, n_2, p_2) and (l_3, m_3, n_3, p_3)

the angle θ between the line (l, m, n, p) and the hyperplane is obtained by the determinant equation *

$$\begin{vmatrix} 1 & \Sigma l_1 l_2 & \Sigma l_1 l_3 & \Sigma l_1 l \\ \Sigma l_1 l_2 & 1 & \Sigma l_2 l_3 & \Sigma l_2 l \\ \Sigma l_1 l_3 & \Sigma l_2 l_3 & 1 & \Sigma l_3 l \\ \Sigma l_1 l & \Sigma l_2 l & \Sigma l_3 l & \cos^2\theta \end{vmatrix} = 0 \qquad (67.2$$

which may also be written in the form—

$$\sin^2\theta \begin{vmatrix} 1 & \Sigma l_1 l_2 & \Sigma l_1 l_3 \\ \Sigma l_1 l_2 & 1 & \Sigma l_2 l_3 \\ \Sigma l_1 l_3 & \Sigma l_2 l_3 & 1 \end{vmatrix} = \begin{vmatrix} l & m & n & p \\ l_1 & m_1 & n_1 & p_1 \\ l_2 & m_2 & n_2 & p_2 \\ l_3 & m_3 & n_3 & p_3 \end{vmatrix}^2.$$

Cor.: The line will be perpendicular to the hyperplane, if $\theta=\pi/2$, or $\phi=0$; *i.e.*, if

$$l : m : n : p = l' : m' : n' : p'.$$

68. Theorem: *All lines perpendicular to a given line at a given point on it lie in a single hyperplane.*

Let l, m, n, p be the direction-cosines of a given line through O. Let (x, y, z, w) be the co-ordinates of a point P on a line through O, at right angles to the given line.

If, then, O is taken as origin, the direction-cosines of OP are proportional to x, y, z, w, and if OP is at right angles to the given line (l, m, n, p), we must have

$$lx + my + nz + pw = 0 \qquad \dots \quad (1)$$

* See author's Analytical Geometry of Hyperspaces, Vol. I, § 25.

which is the locus of the point (x, y, z, w) and represents a unique hyperplane, having the given line as its normal.

Hence, the locus of OP is a single hyperplane.

Suppose the line (l, m, n, p) is perpendicular at any of its points to each of three non-coplanar lines (l_i, m_i, n_i, p_i) $(i=1, 2, 3)$.

Then $$ll_i + mm_i + nn_i + pp_i = 0$$

$$(i=1, 2, 3). \quad \ldots \quad (2)$$

Any direction through the common point, lying in the hyperplane of the three lines, may be taken as

$$al_1 + bl_2 + cl_3, \quad am_1 + bm_2 + cm_3,$$

$$an_1 + bn_2 + cn_3, \quad ap_1 + bp_2 + cp_3.$$

Then, in virtue of the relations (2), we have

$$\Sigma l(al_1 + bl_2 + cl_3) = a\Sigma ll_1 + b\Sigma ll_2 + c\Sigma ll_3 = 0,$$

showing that the given line (l, m, n, p) is perpendicular to any line lying in the hyperplane determined by the three non-coplanar lines.

Hence, *if a line is perpendicular to any three non-coplanar lines at any of its points, the locus of all lines perpendicular to the given line is the hyperplane determined by the three lines.*

Cor. 1: At any point of a line or through any point outside of a line, there can be drawn one and only one hyperplane perpendicular to the line

Cor. 2: Any two of the lines in the hyperplane determine a plane to which the normal to the hyperplane is perpendicular. Hence, a normal to a hyperplane

is perpendicular to all planes of the hyperplane passing through the foot of the normal ; and conversely, every plane perpendicular to a line at a point lies in the hyperplane at right angles to the line.

69. Inclination of a Plane to a Hyperplane :

We have seen in § 52 that the projection of a plane upon a hyperplane is a plane lying in the same hyperplane with the given plane. Consequently, the given plane intersects its projection along the line in which it meets the hyperplane.

From the geometry of the configuration, we at once recognise that the plane makes a fixed angle with the normal direction of the hyperplane, which we can easily determine by the method of ordinary geometry. If, then, ϕ is this angle, the inclination of the plane to the hyperplane may be taken as $\frac{1}{2}\pi - \phi$, which is then a fixed angle.

Following the usual method, the required inclination may be determined by taking a line in the plane and a line in the hyperplane, and then ascertaining the stationary value of the angle between the two lines.

Let the hyperplane and the plane be, as in §52, defined by the equations (1), (2) and (3). Take a direction $(\lambda, \mu, \nu, \rho)$ in the plane and (L, M, N, P) any direction in the hyperplane, the angle between them being θ.

Then, we may take

$$\lambda = \alpha l' + \beta l'', \qquad \mu = \alpha m' + \beta m'',$$

$$\nu = \alpha n' + \beta n'', \qquad \rho = \alpha p' + \beta p''$$

Also

$$\mathrm{L}l + \mathrm{M}m + \mathrm{N}n + \mathrm{P}p = 0 \qquad \ldots \quad (1)$$

subject to the conditions

$$\Sigma\lambda^2 = \Sigma(\alpha l' + \beta l'')^2 = \alpha^2 + \beta^2 + 2\alpha\beta\Sigma l'l'' = 1 \qquad \ldots \quad (2)$$

and
$$L^2 + M^2 + N^2 + P_u^2 = 1. \qquad \ldots \quad (3)$$

If the two chosen directions are along the common line of the plane and the hyperplane, the angle θ vanishes. If the direction (L, M, N, P) be chosen perpendicular to the plane (which is always possible, as this line is then orthogonal to the two lines in the plane and the normal to the hyperplane) then the value of θ becomes $\pi/2$. Besides these two extreme cases, a stationary value of θ will have to be determined.

We have
$$\cos\theta = \Sigma\lambda L = \Sigma(\alpha l' + \beta l'')L$$
$$= \alpha\Sigma l'L + \beta\Sigma l''L \qquad \ldots \quad (4)$$

$$\Sigma Ll = 0, \qquad \Sigma L^2 = 1 \quad \text{and} \quad \alpha^2 + \beta^2 + 2\alpha\beta\Sigma l'l'' = 1.$$

Here, α, β, L, M, N, P are variable parameters. To determine the minimum value of θ, we obtain the critical equations from (4) and (2)

$$\Sigma l'L.\delta\alpha + \Sigma l''L.\delta\beta = 0$$

$$(\alpha + \beta\Sigma l'l'')\delta\alpha + (\beta + \alpha\Sigma l'l'')\delta\beta = 0$$

whence
$$\left.\begin{aligned} \Sigma l'L &= k(\alpha + \beta\Sigma l'l'') \\ \Sigma l''L &= k(\beta + \alpha\Sigma l'l'') \end{aligned}\right\} \qquad \ldots \quad (5)$$

Also, from (1), (3) and (4), varying L, M, N, P, we obtain

$$\lambda = k'l + k''L, \qquad \mu = k'm + k''M,$$
$$\nu = k'n + k''N, \qquad \rho = k'p + k''P \qquad \ldots \quad (6)$$

where k, k', k'' are indeterminate parameters.

Equations (6) show that the plane of the two assumed lines pass through the normal to the hyperplane, and in this position, the angle included between them will give a stationary value ϕ for θ.

Multiplying equations (5) respectively by α and β, and adding, we get

$$k=\alpha\Sigma l'\mathrm{L}+\beta\Sigma l''\mathrm{L}=\cos\phi. \qquad \ldots \quad (7)$$

Multiplying equations (6) respectively by L, M, N, P and adding, we get

$$\Sigma\lambda\mathrm{L}=k'\Sigma\mathrm{L}l+k''\,\Sigma\mathrm{L}^2=k'',\ \textit{i.e.},\ k''=\cos\phi. \quad \ldots \quad (8)$$

Also, squaring and adding equations (6), we get

$$\begin{aligned}\Sigma\lambda^2=1&=k'^2\Sigma l^2+k''^2\Sigma\mathrm{L}^2+2k'k''\,\Sigma\mathrm{L}l\\ &=k'^2+k''^2. \qquad \ldots \quad (9)\end{aligned}$$

From (8) and (9), we have then

$$\begin{aligned}k'^2=1-k''^2&=1-\cos^2\phi\\ &=\sin^2\phi.\end{aligned}$$

Hence, $\lambda=l\sin\phi+\mathrm{L}\cos\phi,\quad \mu=m\sin\phi+\mathrm{M}\cos\phi,$

$\nu=n\sin\phi+\mathrm{N}\cos\phi,\quad \rho=p\sin\phi+\mathrm{P}\cos\phi.$

Eliminating $\sin\phi$ and $\cos\phi$, we get

$$\begin{Vmatrix}\lambda & \mu & \nu & \rho\\ l & m & n & p\\ \mathrm{L} & \mathrm{M} & \mathrm{N} & \mathrm{P}\end{Vmatrix}=0 \qquad (69.1$$

showing that the three lines are coplanar.

Multiplying equations (6) by l', m', n', p' respectively and adding, we get

$$\alpha\Sigma l'^2+\beta\Sigma l'l''=\sin\phi\,\Sigma ll'+\cos\phi\,\Sigma\mathrm{L}l'$$

or, $$\alpha+\beta\Sigma l'l''=\sin\phi\ \Sigma ll'+\cos\phi\ \Sigma Ll'. \quad \dots \quad (10)$$

Similarly, multiplying the same equations by l'', m'', n'', p'', and adding, we get

$$\alpha\Sigma l'l''+\beta\Sigma l''^2=\sin\phi\ \Sigma ll''+\cos\phi\ \Sigma Ll''$$

or, $$\alpha\Sigma l'l''+\beta=\sin\phi\ \Sigma ll''+\cos\phi\ \Sigma Ll''. \quad \dots \quad (11)$$

From (5) and (10) we obtain

$$\alpha+\beta\Sigma l'l''=\sin\phi.\ \Sigma ll'+\cos\phi\ (\alpha+\beta\Sigma l'l'')\cos\phi$$

$$=\sin\phi\ \Sigma ll'+(\alpha+\beta\Sigma l'l'')\cos^2\phi$$

$$\left.\begin{array}{ll}\text{whence} & (\alpha+\beta\Sigma l'l'')\sin\phi=\Sigma ll'. \\ \text{Similarly,} & (\beta+\alpha\Sigma l'l'')\sin\phi=\Sigma ll''.\end{array}\right\} \quad \dots \quad (12)$$

By means of relations (12), eliminating α and β from the relation $\alpha^2+\beta^2+2\alpha\beta\ \Sigma\ l'l''=1$, we obtain

$$\sin^2\phi\,\{1-(\Sigma l'l'')^2\}=(\Sigma ll')^2+(\Sigma ll'')^2-2\Sigma ll'.\ \Sigma ll''.\ \Sigma l'l''. \quad (69.2$$

Thus, the critical angle between the lines is given by this equation (69.2).

If α', β', γ' be the angles between the three lines, namely, the normal to the hyperplane and the two lines in the plane, we have

$$\cos\alpha'=\Sigma ll', \qquad \cos\beta'=\Sigma ll'', \qquad \cos\gamma'=\Sigma l'l''.$$

The above formula can then be written as

$$\sin^2\phi.\ \sin^2\gamma'=\cos^2\alpha'+\cos^2\beta'-2\cos\alpha'\cos\beta'\cos\gamma'.$$

$$\sin^2\phi\ \sin^2\gamma'+\cos^2\gamma'+1=1+\cos^2\alpha'+\cos^2\beta'+\cos^2\gamma'$$
$$-2\cos\alpha'\cos\beta'\cos\gamma'. \quad (69.3$$

The plane of the two critical lines passes through the normal of the hyperplane, and this plane is perpendicular to the line in which the given plane meets the hyperplane. For, if (l_1, m_1, n_1, p_1) be the common line, we have

$$l_1 = a_1 l' + \beta_1 l'', \quad m_1 = a_1 m' + \beta_1 m'',$$

$$n_1 = a_1 n' + \beta_1 n'', \quad p_1 = a_1 p' + \beta_1 p''$$

where $$a_1{}^2 + \beta_1{}^2 + 2a_1\beta_1 \Sigma l'l'' = 1.$$

Also, $$\Sigma l(a_1 l' + \beta_1 l'') = a_1 \Sigma ll' + \beta_1 \Sigma ll'' = 0$$

$$\therefore \quad \frac{a_1}{\Sigma ll''} = \frac{\beta_1}{-\Sigma ll'} = \frac{1}{\{(\Sigma ll'')^2 + (\Sigma ll')^2 - 2\Sigma ll' \Sigma ll'' \Sigma l'l''\}^{\frac{1}{2}}}$$

$$= \frac{1}{\sin\phi \,.\, \sin\gamma'}. \tag{69.4}$$

Now, $\Sigma \lambda l_1 = a_1 \Sigma \lambda l' + \beta \Sigma \lambda l''$

$$= \sin\phi (a_1 \cos a' + \beta_1 \cos\beta') + \cos\phi (a_1 \Sigma Ll' + \beta_1 \Sigma Ll'')$$

$$= \cos\phi . \frac{1}{\sin\phi . \sin\gamma'} (\Sigma ll'' \,.\, \Sigma Ll' - \Sigma ll' \,.\, \Sigma Ll'')$$

$= 0$, in virtue of relations (5) and (12).

Similarly, $\Sigma Ll_1 = 0$, showing that the direction (l_1, m_1, n_1, p_1) is orthogonal to both $(\lambda, \mu, \nu, \rho)$ and (L, M, N, P), and is therefore perpendicular to the plane.

Cor.: If, then, $\phi = \pi/2$, the plane is said to be *perpendicular* to the hyperplane. In this case,

$$\Sigma \lambda l = \sin\phi \, \Sigma l^2 + \cos\phi \, \Sigma Ll$$

$$= \sin\phi = \sin \tfrac{1}{2}\pi = 1.$$

Hence, the critical line $(\lambda, \mu, \nu, \rho)$ coincides with the normal to the hyperplane, and the given plane contains the normal direction of the hyperplane. It will be noticed further that the normal to the plane at any common point always lies in the hyperplane, *i.e.*, the absolutely perpendicular plane at that point lies in the hyperplane.

We therefore define that *a plane is orthogonal to a hyperplane, when it contains the normal direction of the hyperplane, and the absolutely perpendicular plane at any common point lies in the hyperplane.*

Ex. The plane $\Sigma l'x - k' = 0$, $\Sigma l''x - k'' = 0$ is projected on the hyperplane $\Sigma lx - k = 0$.

The projecting hyperplane is $(\Sigma l'x - k') + \lambda\,(\Sigma l''x - k'') = 0$. If this is to be orthogonal to the given hyperplane, we must have

$$\Sigma l(l' + \lambda l'') = 0, \quad \text{or,} \quad \Sigma ll' + \lambda \Sigma ll'' = 0.$$

Eliminating λ, the projecting hyperplane is given by

$$\frac{\Sigma l'x - k'}{\Sigma\; ll'} = \frac{\Sigma l''x - k''}{\Sigma ll''}.$$

70. Theorem: *If a plane is perpendicular to a hyperplane, any plane in the hyperplane perpendicular to their common line is absolutely perpendicular to the plane, and any plane absolutely perpendicular to the plane through any point of the hyperplane lies entirely in the hyperplane.*

Without loss of generality, we may take the equations of the hyperplane and the plane perpendicular to it respectively in the forms

$$Lx + My + Nz + Pw = 0 \qquad \dots \quad (1)$$

$$\begin{vmatrix} x & y & z & w \\ L & M & N & P \\ l & m & n & p \end{vmatrix} = 0 \qquad \dots \quad (2)$$

where (l, m, n, p) is the common direction lying in the plane and the hyperplane. Take λ_i and μ_i two directions through the origin, lying in the hyperplane and perpendicular to the direction (l, m, n, p).

Then, $\Sigma\lambda L = 0$, $\Sigma\mu L = 0$, $\Sigma\lambda l = 0$, $\Sigma\mu l = 0$.

These equations show that the lines λ and μ are each perpendicular to the two guiding lines of the plane (2), and consequently, they are normals to the plane (2). Hence, the plane determined by λ and μ is absolutely perpendicular to the plane (2).

If, on the other hand, the plane determined by λ and μ is absolutely perpendicular to the plane (2), we must have

$$\Sigma L\lambda = 0, \quad \Sigma L\mu = 0 \quad \text{and} \quad \Sigma\lambda l = 0, \quad \Sigma\mu l = 0.$$

The first two relations show that the lines λ and μ are each at right angles to the normal to the hyperplane and are, therefore, contained therein. Hence, the plane of λ and μ lies in the hyperplane.

71. Theorem: *Two hyperplanes perpendicular to a plane at any of its points intersect in the plane absolutely perpendicular to the plane at that point.*

Let $Lx + My + Nz + Pw = 0$ and $L'x + M'y + N'z + P'w = 0$ be two hyperplanes perpendicular to a plane at the origin.

Since the plane is perpendicular to the hyperplanes, it contains the normals to the two hyperplanes, and consequently, its equations are

$$\left\| \begin{matrix} x & y & z & w \\ L & M & N & P \\ L' & M' & N' & P' \end{matrix} \right\| = 0.$$

Evidently then, the plane of intersection of the two hyperplanes is absolutely perpendicular to this plane, which is determined by the two normals of the hyperplanes.

72. Theorem: *If a plane is perpendicular to a hyperplane, any hyperplane which contains the plane is perpendicular to the hyperplane.*

Let $\Sigma lx=0$ and $\Sigma l'x=0$ be the equations of the plane, and let $\Sigma Lx=0$ be that of the hyperplane.

Since the hyperplane is perpendicular to the plane, its normal must lie in the plane, and hence,

$$\Sigma Ll=0, \quad \Sigma Ll'=0. \qquad \dots \quad (1)$$

Any hyperplane containing the plane is

$$\Sigma lx + k\Sigma l'x = 0.$$

This will be perpendicular to the given hyperplane, if $\Sigma L(l+kl')=0$, *i.e.*, if $\Sigma Ll + k\Sigma Ll'=0$. But, in virtue of the relations (1), this is satisfied, and hence the truth of the theorem follows.

73. Inclination of a Line to a Plane:

In § 55 we have considered the projection of any line upon a given plane. If, then, L, M, N, P be the direction-cosines of the line and L′, M′, N′, P′ those of its projection upon the given plane (2) (§ 55), then

$$\cos\theta = LL' + MM' + NN' + PP',$$

where θ is the angle between the line and its projection.

This angle θ is defined to be the angle between the line and the plane. Since (L′, M′, N′, P′) is a direction in the plane (2) and the projecting hyperplane (55.1), we have

$$\Sigma L'l = 0, \qquad \Sigma L'm = 0 \tag{73.1}$$

and

$$\begin{vmatrix} L' & M' & N' & P' \\ L & M & N & P \\ l_1 & l_2 & l_3 & l_4 \\ m_1 & m_2 & m_3 & m_4 \end{vmatrix} = 0.$$

From these three equations the ratios L′ : M′ : N′ : P′, and consequently, their values can be determined, whence the value of $\cos\theta$ can be calculated. If, however, the plane be defined by its guiding lines λ_i, μ_i, the value of θ may be obtained in a simpler form.

Let L′, M′, N′, P′ be a direction in the plane, so that we may take

$$L' = p\lambda_1, + q\mu_1, \quad M' = p\lambda_2 + q\mu_2,$$

$$N' = p\lambda_3 + q\mu_3, \quad P' = p\lambda_4 + q\mu_4$$

where $L'^2 + M'^2 + N'^2 + P'^2 = p^2 + q^2 + 2pq\ (\lambda\mu) = 1$....(1)

In order to determine a minimum value of θ, it will be noticed that θ cannot be zero, nor $\pi/2$, *i.e.*, $0 \leq \theta \leq \pi/2$. We have

$$\cos\theta = \mathrm{LL}' + \mathrm{MM}' + \mathrm{NN}' + \mathrm{PP}'$$

$$= \Sigma \mathrm{L}\ (p\lambda_1 + q\mu_1) = p\ (\mathrm{L}\lambda) + q\ (\mathrm{L}\mu). \quad \ldots \quad (2)$$

Differentiating (1) and (2) with respect to p and q, we obtain

$$\{p + (\lambda\mu)\ q\}\ \delta p + \{q + (\lambda\mu)\ p\}\ \delta q = 0$$

$$(\mathrm{L}\lambda)\ \delta p + (\mathrm{L}\mu)\ \delta q = 0$$

$$\therefore \quad p + (\lambda\mu)\ q = k\ (\mathrm{L}\lambda)$$

$$q + (\lambda\mu)\ p = k\ (\mathrm{L}\mu).$$

Multiplying these respectively by p and q, and adding, we get

$$k\cos\theta = 1, \quad i.e., \quad k = 1/\cos\theta.$$

$$\therefore \quad \cos\theta\ \{\ p + q\ (\lambda\mu)\ \} = (\mathrm{L}\lambda)$$

$$\cos\theta\ \{\ p\ (\lambda\mu) + q\ \} = (\mathrm{L}\mu).$$

Also, $$p\ (\mathrm{L}\lambda) + q\ (\mathrm{L}\mu) = \cos\theta.$$

Eliminating p and q, we get

$$\begin{vmatrix} \cos\theta & \cos\theta\ (\lambda\mu) & (\mathrm{L}\lambda) \\ \cos\theta\ (\lambda\mu) & \cos\theta & (\mathrm{L}\mu) \\ (\mathrm{L}\lambda) & (\mathrm{L}\mu) & \cos\theta \end{vmatrix} = 0$$

whence
$$\begin{vmatrix} 1 & (\lambda\mu) & (L\lambda) \\ (\lambda\mu) & 1 & (L\mu) \\ (L\lambda) & (L\mu) & \cos^2\theta \end{vmatrix} = 0 \qquad (73.2$$

$$\text{or, } \sin^2\theta\,[\lambda\mu]^{2*} = \begin{vmatrix} 1 & (\lambda\mu) & (L\lambda) \\ (\lambda\mu) & 1 & (L\mu) \\ (L\lambda) & (L\mu) & 1 \end{vmatrix}$$

$$i.e., \sin^2\theta.[\lambda\mu]^2 = 1-(L\lambda)^2-(L\mu)^2-(\lambda\mu)^2+2(L\lambda)(L\mu)(\lambda\mu) \qquad (73.3$$

Cor. : If $\theta = 90°$, we have

$$(L\lambda)^2+(L\mu)^2 = 2(L\lambda)(L\mu)(\lambda\mu).$$

74. Inclination of two Planes :

The consideration of the inclination of any two planes presents considerable difficulties, as the planes intersect in only one point—a phenomenon peculiar to the fourfold, and it is not possible to visualise the configuration. The inclination of two geometrical entities is generally defined as the amount of rotation of one around their common locus, so as to coincide with the other. In the case of two planes, the inclination may analogously be defined as the operation (whatever its nature) or operations, which brings the two planes into coincidence, and a conventional measure of this operation may be taken as a measure of the inclination of the two planes. It is, therefore, necessary at the outset to determine the operation or operations which

* Analytical Geometry of Hyperspaces, Vol. I, § 25.

bring the two planes into coincidence. It may be pointed out, however, that the operation will not generally be a simple one.

Various methods, both analytic and geometric, have been suggested for ascertaining this operation. We shall indicate, at the beginning, a geometric method which will go a great way in giving some idea with regard to the general treatment in analysis.

75. The Method of Projection:

Consider two planes α and β meeting in the common point O. If we take a line l in α and a line m in β, there is always a plane angle θ between l and m. There is a singly infinite system of lines l in α and a singly infinite system of lines m in β. Therefore, the angle θ between any two such lines may be regarded as a function of two independent variables, each admitting values from $-\infty$ to $+\infty$. In fact, there is a one-to-one correspondence between the lines l and m; or what is the same thing, we may consider a pencil of lines l in α, homographically related to the lines m in β. If now the lines l in α are projected into the lines l' in β, in the plane β we obtain two superposed pencils l' and m, with the common vertex O, homographically related to each other. There are then always two self-corresponding rays of the pencils,* and they are, therefore, such that each self-corresponding ray is the projection of the corresponding ray in the other plane α, and is itself projected into the same; or, in other words,

* L. Cremona, *Elements of Projective Geometry*, p. 68, § 82.

if a_1 be a critical line in α and b_1 its projection on β, then b_1 is the critical line of β and a_1 is its projection on α. Thus, the two lines a_1 and b_1 are the projections of each other.

Let a_1, a_2 be the critical lines in α and b_1, b_2 those in β. It is now easy to conclude, therefore, that the plane determined by a_1, b_1 intersects α and β both orthogonally, and similarly, also the plane determined by a_2, b_2. It is then clearly seen that, if a series of planes be drawn to intersect the two given planes α and β in lines, there are always two, which intersect them both orthogonally, *i.e.*, there are always two common perpendicular planes of α and β.

Let ϕ and ψ be the two critical angles cut out by α and β on these two common perpendicular planes. Thus, it is found that among the angles cut out on the ∞ system of planes by α and β, there are two, and only two, whose planes are common perpendicular to α and β. It is also known that these two angles are always real * and unique, and they are invariant for the two given planes. † The inclination of α and β can, therefore, be defined by means of these two invariant angles. Some have defined these two critical angles as the angles between the two planes. If the notion of a higher dimensional angle is to be introduced, it can safely be assumed that the inclination of the two planes is characterised by the four critical lines a_1, a_2, b_1, b_2 ; and the two planes α and β form between them the "tetrahedroidal angle" of which the four edges are a_1, b_1, a_2, b_2 ; the six face-

* Castelnuovo, *Atti del R. Istituto Veneto* (1885).

† C. Jordan, *loc. cit.* §32.

angles are the angles $\hat{a_1 b_1}$, $\hat{a_1 b_2}$, $\hat{a_1 a_2}$, $\hat{b_1 b_2}$, $\hat{b_1 a_2}$, $\hat{a_2 b_2}$ and the four *trihedral* angles are $a_1 a_2 b_1$, $a_1 a_2 b_2$, $a_1 b_1 b_2$ and $a_2 b_1 b_2$. Since the four critical lines are invariant in position, the tetrahedroidal angle defined by means of these lines is also invariant and may be taken to measure the inclination of the two planes α and β.

76. The Tetrahedroidal Angle :

The notion of a tetrahedroidal angle is a mere abstraction and will not carry us further in realising the actual operations which bring the planes α and β into coincidence. For this purpose, it is necessary, therefore, to analyse the nature of this angle and to establish its invariant character.

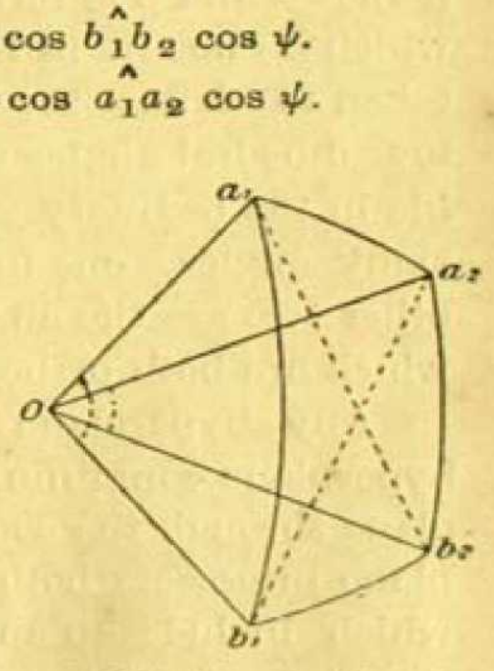
It is easy to see that the critical lines in each plane are mutually orthogonal. For, consider the two rectangular trihedral angles $a_1 b_1 a_2$ and $a_2 b_2 b_1$. We have

$$\cos \hat{a_2 b_1} = \cos \hat{a_1 a_2} \cos \phi = \cos \hat{b_1 b_2} \cos \psi.$$

Again, $\cos \hat{a_1 b_2} = \cos \hat{b_1 b_2} \cos \phi = \cos \hat{a_1 a_2} \cos \psi.$

It follows, therefore, that

$$\cos \hat{a_1 a_2} = \cos \hat{b_1 b_2};$$



and consequently,

since $\cos \phi \neq \cos \psi$,

$$\cos \hat{a_1 a_2} = \cos \hat{b_1 b_2} = 0$$

So also,

$$\cos \hat{a_2 b_1} = \cos \hat{a_1 b_2} = 0$$

i.e., $\hat{a_1 a_2}$, $\hat{b_1 b_2}$, $\hat{a_1 b_2}$, $\hat{a_2 b_1}$ are each a right angle.

Thus, it is found that the plane $a_1 O b_1$ is absolutely perpendicular to the plane $a_2 O b_2$, *i.e.*, the critical planes

are mutually absolutely perpendicular, and it is evident that the two double rays of each of the pencils are mutually perpendicular.

The lines a_1, a_2 are respectively the projections of b_1 and b_2, and *vice versa*. The relation between the two planes is, therefore, reciprocal. We have thus obtained the two angles ϕ and ψ, fixed in position and magnitude, and they are invariant for the two planes (except as to signs, which will be considered later). These two angles then characterise the inclination of two planes which intersect in a point, and we say that two pairs of planes are congruent, when these angles are the same. These two angles, therefore, are intrinsically related to the tetrahedroidal angle, and may be said to form the principal constituents of the same.

77. The Maximum and Minimum Angles:

The 'tetrahedroidal angle' conceived between two planes α and β is formed by the four lines a_1, a_2, b_1, b_2, which do not lie in the same hyperplane. This may be taken to define the inclination of the two planes. Its four dihedral angles are right angles, which are impossible in the ordinary space. Of its face-angles, two are right angles, one in each of the planes α and β. The other two are the angles ϕ and ψ in the critical planes, which are both orthogonal to α and β.

Any hyperplane containing the plane α and any hyperplane containing the plane β intersect in a third plane through the common point O of α and β. This plane intersects both the given planes α and β in lines, which include an angle between them and this angle has, consequently, a maximum and a minimum value, which are the two angles ϕ and ψ; and in their critical

positions, their planes intersect the two given planes both orthogonally.

In the plane α draw a line a and in β draw a line b. If now b is allowed to remain fixed in position, while a performs a complete rotation in the plane α, the angle $b\hat{a}$ has a minimum value θ, which is called the angle between the line 'b' and the plane α. This angle is the angle which b forms with its projection on the plane α. If now b is again supposed to rotate about O, in the plane β, then the angle θ varies and attains a maximum and a minimum value, which are ϕ and ψ. This property of the angles ϕ and ψ enables us to define the angles between two planes in the fourfold as the maximum and minimum values of the angles which a line in one plane forms with a line in the other. This is the definition adopted by P. Schoute,* and since the double lines, *i.e.*, the critical lines are real, the angles ϕ and ψ are also real.

It is now clearly seen that any two planes α and β have two, and only two, common perpendicular planes intersecting them both in lines. There can be no third plane intersecting them both orthogonally, since there can be no plane absolutely perpendicular to both the planes of ϕ and ψ.

78. Let OA_1, OA_2 be the critical lines of α and OB_1, OB_2 those of β, so that

$$\angle A_1OB_1 = \psi \quad \text{and} \quad \angle A_2OB_2 = \phi.$$

In α, construct the rectangle $OP_1P'Q_1$, as shown in the diagram, and project it on β by drawing perpendiculars from P_1, Q_1 and P' on β. These perpendiculars

* P. H. Schoute, *Mehrdimensionale Geometrie*, Vol. I.

are not parallel lines, but the projections of parallel lines are parallel. Consequently, if OP_2RQ_2 is the projection, OP_2 is parallel to Q_2R and OQ_2 is parallel to P_2R.

$\therefore$ OP_2RQ_2 is a rectangle, since $\angle B_1OB_2$ is a right angle.

Since OP_2 is the projection of OP_1, we have

$$OP_2 = OP_1 \cos\phi \quad \text{and} \quad OQ_2 = OQ_1 \cos\psi.$$

$$\therefore \quad \frac{OQ_2}{OP_2} = \frac{P_2R}{OP_2} = \frac{OQ_1 \cos\psi}{OP_1 \cos\phi}. \qquad \ldots \quad (1)$$

If now,

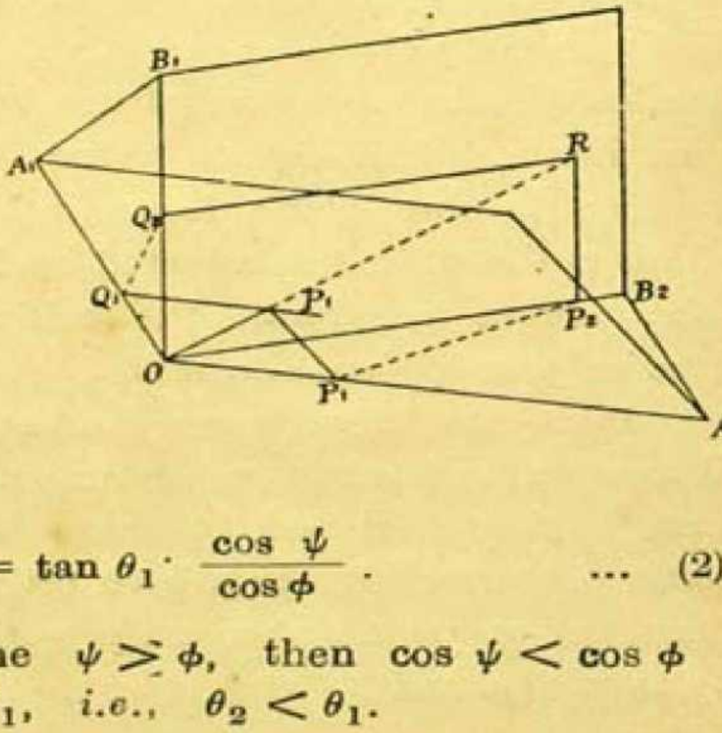
$\angle P_1OP' = \theta_1$, and

$\angle P_2OR = \theta_2$,

we have

$$\frac{P_1P'}{OP_1} = \tan\theta_1$$

and $\dfrac{P_2R}{OP_2} = \tan\theta_2$,



whence

$$\frac{OQ_2}{OP_2} = \tan\theta_2 = \tan\theta_1 \cdot \frac{\cos\psi}{\cos\phi}. \qquad \ldots \quad (2)$$

If now, we assume $\psi \gtrless \phi$, then $\cos\psi < \cos\phi$ and $\therefore \tan\theta_2 < \tan\theta_1$, *i.e.*, $\theta_2 < \theta_1$.

This shows that, if the two rectangles are placed one upon the other, the diagonal OR falls within the angle P_1OP'.

If, again, the rectangle OP_2RQ_2 is projected back on the plane α, and OP_3 is the projection of OP_2, and so on, then

$$OP_3 = OP_2 \cos\phi, \quad OQ_3 = OQ_2 \cos\psi.$$

$$\therefore \quad \frac{OQ_3}{OP_3} = \tan \theta_3 = \frac{OQ_2}{OP_2} \cdot \frac{\cos \psi}{\cos \phi} = \tan \theta_2 \cdot \frac{\cos \psi}{\cos \phi}$$

$$= \tan \theta_1 \cdot \left(\frac{\cos \psi}{\cos \phi} \right)^2 ; \text{ and so on.} \qquad (78.1$$

These show that when OR, the projection on β, of any line OP′ in α is projected back again on to α, OP′ is no longer the projection of OR, but it is some other line OR_3 different from OP′, since $\theta_3 < \theta_2 < \theta_1$. OR_3 may coincide with OP′, only when $\theta_1 = \theta_3$, and this is possible only in two cases, namely, when

$$\tan \theta_1 = 0 \quad \text{or} \quad \infty .$$

Case I: When $\tan \theta_1 = 0$, $\theta_1 = 0 = \theta_3$, and the line OP′ coincides with OA_2 and OA_2, OB_2 become the projections of each other.

Case II: When $\tan \theta_1 = \infty$, $\theta_1 = 90°$, and OP′ coincides with OA_1, and OA_1, OB_1 become the projections of each other.

79. The Nature of the Angles ϕ and ψ:

From what has been said above, we are now in a position to recognise the true nature of the angles ϕ and ψ. One of the angles, say ϕ, is necessarily the real minimum angle—an absolute minimum. What then is the other angle ψ ? Is it the absolute maximum—the greatest of the angles which a line of α may form with a line in β ? It will be clearly seen that the maximum value of ψ can never be greater than 90°; for, two lines at O form two acute and two obtuse angles, but then the obtuse angle has no meaning, because

there is the supplementary acute angle to be taken into account. Thus, it is clear that the absolute maximum is a right angle. In fact, if OP is any line in α, all its perpendicular lines at O lie in a hyperplane, which intersects β in a line through O, and this line forms with OP a right angle. To each line of α, then, there corresponds a line in β through O which is perpendicular to it. From this, the real nature of ϕ and ψ can easily be recognised. Of all the angles, those are the least whose arms are the projections of each other ; and of these, ψ is the greatest and ϕ is the least. ψ, therefore, may be called a " relative maximum."

From the above formulae, it is at once evident that the values of θ_1, θ_2, θ_3,.........gradually diminish ; and if we proceed indefinitely, the two lines OP′ and OR gradually approach the arms of ϕ, *i.e.*, ϕ is the minimum of these angles.

Writing the above equations in the reverse order, we get

$$\tan \theta_1 = \tan \theta_2 \cdot \frac{\cos \phi}{\cos \psi},$$

$$\tan \theta_1 \cdot \frac{\cos \phi}{\cos \psi} = \tan \theta_2 \cdot \left(\frac{\cos \phi}{\cos \psi} \right)^2, \text{ and so on.} \qquad (79.1$$

Here, we consider the angles whose one arm is the projection of the other, and not conversely.

In these cases, θ_1, θ_2, θ_3......gradually increase in value, and therefore, the two arms gradually approach those of ψ. If ϕ is the least, starting from ϕ, the arms of the angles gradually move away from each other, so that ψ is the maximum value.

The preceding considerations suggest that, if a circle with O as centre be described in the plane α, its projection on the plane β is an ellipse. For, we have

$$OP_2 = OP_1 \cos\phi, \quad OQ_2 = OQ_1 \cos\psi.$$

$$\therefore \; OP_1 = \frac{OP_2}{\cos\phi}, \quad OQ_1 = \frac{OQ_2}{\cos\psi}.$$

$$\text{If } OP' = r, \text{ then } OP_1{}^2 + OQ_1{}^2 = OP'^2 = r^2. \quad (79.2$$

Hence, the locus of P′ is always a circle, and we have

$$\frac{OP_2{}^2}{\cos^2\phi} + \frac{OQ_2{}^2}{\cos^2\psi} = r^2, \text{ or } \frac{OP_2{}^2}{r^2\cos^2\phi} + \frac{OQ_2{}^2}{r^2\cos^2\psi} = 1 \quad (79.3$$

which represents an ellipse in the plane β, shewing that R, the projection of P′ on β describes an ellipse with semi-axes $r\cos\phi$ and $r\cos\psi$, $r\cos\phi$ lying along OB_2, and $r\cos\psi$ along OB_1. If now $r=1$, the variation of the angle which OP′ makes with its projection OR with the variation of OP′ in α is manifest.

Since $\phi < \psi$, $\cos\phi > \cos\psi$; and since $\cos\phi$ is the semi-major axis, $\cos\psi$ the semi-minor axis of the ellipse, starting with the least angle ϕ, it is found that as the radius OP′ moves round, the angle ϕ gradually increases till it attains the maximum value ψ ; again it decreases to ϕ and then again increases to ψ, and then finally decreases to ϕ.

80. Normally associated Elements at Infinity :

The preceding results can be established with the help of the *normally associated* elements at infinity. A system of parallel lines have the same point L_∞ at infinity, and the hyperplanes perpendicular to these lines are themselves parallel and have a common plane λ_∞ at

infinity. Thus, the points and planes of the hyperplane at infinity can be associated in such a manner that each line through any point L_∞ is perpendicular to each hyperplane containing any plane λ_∞, and conversely. The point L_∞ and the plane λ_∞ are then said to be "normally associated." Similarly, each point of λ_∞ is normally associated with L_∞, since each line through this point is perpendicular to each line through L_∞, and each line of λ_∞ is normally associated with L_∞, since each plane through this line is perpendicular to each line through L_∞. Similarly, each line of λ_∞ is normally associated with each line through L_∞, since each plane through this line is perpendicular to each line through L_∞.

Two lines a_∞ and a^*_∞ at infinity may be said to be "normally associated," when each plane through one line is absolutely perpendicular to each plane through the other. In fact, if a_∞ is the line at infinity in any plane α, then a^*_∞ is the line at infinity in its absolutely perpendicular plane α', *i.e.*, a^*_∞ is the line in which the plane α' meets the hyperplane at infinity. These are, in fact, pole and polar elements with reference to the *Absolute* in the fourfold.*

81. The Existence of two invariant Angles:

That there always exist two invariant angles between two planes in general position in the fourfold can also be demonstrated with the help of *normally associated* elements at infinity. Let the two planes

* Cayley, *Collected Works*, Vol. II, pp. 583-592; Vol. XIII, pp. 41-42 and pp. 480-504. The notion can be extended to the fourfold.

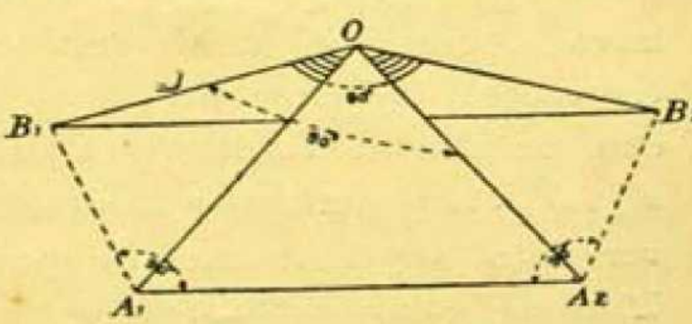
α and β intersect in the point O, and let a_∞ and b_∞ be their two lines at infinity respectively, and a^*_∞, b^*_∞ be their normally associated lines. The planes α and β being arbitrary, the four lines a_∞, b_∞, a^*_∞, b^*_∞ have general positions and are, therefore, non-intersecting. If γ is a common orthogonal plane of α and β, then γ intersects the planes α and β as well as their absolutely perpendicular planes α' and β' in lines ; consequently, γ intersects the four lines a_∞, b_∞, a^*_∞, b^*_∞ at infinity, *i.e.*, the four lines have a common transversal at infinity. In order, therefore, to ascertain how many planes, such as γ, can be drawn to intersect all the four lines a_∞, b_∞, a^*_∞, b^*_∞, it is necessary to determine the common transversals of any four lines in space in general position. This is easily obtained by using the properties of generating lines of a ruled surface. Three of the four lines may be taken to generate a ruled surface of the second degree : the fourth line intersects it in two points. The two generators of the opposite system, which may be drawn through these two points, will intersect those three generators, and consequently all the four lines. Thus, there are two common transversals of the four non-intersecting lines, and these two, with the common point O, determine two planes which intersect α and β orthogonally, and on these two α and β cut out the two angles.

The existence of these two angles between α and β may also be established in another manner :—

Since the two planes have a common point O, we can draw a line 'a' in α and a line 'b' in β. These lines will include an angle θ, which has a minimum

value—an absolute minimum. Denoting this minimum value by ϕ, it can easily be seen that the two arms of ϕ are, one the projection of the other. The plane of ϕ then contains the normals to both α and β at O, and is, therefore, orthogonal to both, so that the line at infinity of the plane of ϕ intersects all the four lines a_{∞}, b_{∞}, a^*_{∞}, b^*_{∞}.

Let OA_1 (in α) and OB_1 (in β) be the two arms of ϕ. Draw OA_2 in α a perpendicular on OA_1 and OB_2 in β perpendicular on OB_1. We shall now prove that the plane OA_2B_2 intersects both α and β orthogonally, and therefore, the angle $A'_2OB_2 \equiv \psi$ is the second angle between α and β, and that the plane of ψ is absolutely perpendicular to that of ϕ.



Since α is orthogonal to the plane of ϕ and OA_2 is perpendicular to OA_1, it follows that OA_2 is perpendicular to the plane of ϕ.

$$\therefore \quad \angle A_2OB_1 = 90°.$$

Similarly, OB_2 is perpendicular to OA_1, so that $\angle A_1OB_2 = 90°$. Since OA_2 is perpendicular to both OA_1 and OB_1 in the plane of ϕ, it is perpendicular to all lines in the same plane. Hence, it follows that the planes of ϕ and ψ are mutually absolutely perpendicular. Again, since the lines in the plane ψ are all perpendicular to OA_1 and OB_1, the plane of ψ is orthogonal to both α and β, just as the plane of ϕ

is. Thus, both the planes of ϕ and ψ stand in the same relation to the two given planes α and β. This, therefore, agrees with the fact already established that there are two angles between any two planes in general position in the fourfold. There is no third plane which is absolutely perpendicular to the planes of ϕ and ψ, and hence there are only two such angles between α and β.

82. Analytical Discussion :

Let $l_i, m_i, p_i, q_i (i=1, 2, 3, 4)$ be the direction-cosines of the four lines OA_1, OA_2, OB_1, OB_2 respectively, referred to a system of rectangular axes. Let λ_i be the direction-cosines of any line in the plane α and μ_i those of a line in β ; and let θ be the angle between the lines λ and μ.

Then, $\lambda_i = Al_i + Bm_i$ and

$$\therefore \quad 1 = \Sigma\lambda_i^2 = A^2+B^2+2AB(lm) \quad \dots \quad (1)$$

$\mu_i = Cp_i + Dq_i$ and

$$\therefore \quad 1 = \Sigma\mu_i^2 = C^2+D^2+2CD(pq) \quad \dots \quad (2)$$

where A, B, C, D are indeterminate multipliers.

Also, $\cos\theta = \Sigma\lambda_i\mu_i = \Sigma(Al_i + Bm_i)(Cp_i + Dq_i)$

$$= AC(lp) + BC(mp) + AD(lq) + BD(mq). \quad \dots (3)$$

For ascertaining the maximum and minimum values of θ, we obtain by differentiating the equations (1), (2) and (3)—

$$0 = \{A + B(lm)\}\delta A + \{B + A(lm)\}\delta B \quad \dots \quad (1')$$

$$0 = \{C + D(pq)\}\delta C + \{D + C(pq)\}\delta D \quad \dots \quad (2')$$

$$0=\{C(lp)+D(lq)\}\delta A+\{C(mp)+D(mq)\}\delta B$$
$$+\{A(lp)+B(mp)\}\delta C+\{A(lq)+B(mq)\}\delta D. \quad \dots \quad (3')$$

Both the lines λ and μ may be varied. When λ is kept fixed, while μ varies, $\delta A=\delta B=0$; when μ is kept fixed, while λ varies, $\delta C=\delta D=0$.

Comparing equations (1′) and (3′) we obtain

$$A+B(lm)=k\{C(lp)+D(lq)\} \quad \dots \quad (4)$$

$$B+A(lm)=k\{C(mp)+D(mq)\} \quad \dots \quad (5)$$

where k is still another indeterminate multiplier.

Similarly, comparing (2′) and (3′), we obtain

$$A(lp)+B(mp)=k'\{C+D(pq)\} \quad \dots \quad (6)$$

$$A(lq)+B(mq)=k'\{D+C(pq)\} \quad \dots \quad (7)$$

where k' is again another multiplier.

Multiplying (4) by A and (5) by B and adding, we obtain

$$k\cos\theta=1, \quad \text{or} \quad k=1/\cos\theta.$$

Similarly, from (6) and (7), we obtain

$$k'=\cos\theta.$$

Now substituting these values of k and k', the equations (4), (5), (6) and (7) may be written as

$$\left.\begin{aligned} A\cos\theta+B\cos\theta\,(lm)-C(lp)-D(lq)&=0\\ A\cos\theta(lm)+B\cos\theta-C(mp)-D(mq)&=0\\ A(lp)+B(mp)-C\cos\theta-D\cos\theta\,(pq)&=0\\ A(lq)+B(mq)-C\cos\theta\,(pq)-D\cos\theta&=0 \end{aligned}\right\} \quad \dots \quad (8)$$

Now eliminating the four parameters A, B, C, D, the critical values of θ satisfy the determinant equation

$$\begin{vmatrix} \cos\theta & \cos\theta(lm) & (lp) & (lq) \\ \cos\theta(lm) & \cos\theta & (mp) & (mq) \\ (lp) & (mp) & \cos\theta & \cos\theta(pq) \\ (lq) & (mq) & \cos\theta(pq) & \cos\theta \end{vmatrix} = 0 \qquad (82.1$$

which may be written in the form—

$$[lm]^2[pq]^2 \cos^4\theta + \mathrm{H}\cos^2\theta + [lm/pq]^2 = 0 \qquad (82.2$$

where H stands for the co-efficient of $\cos^2\theta$ in (82.1), and

$$(lm) \equiv \text{cosine of the angle } \widehat{lm}$$

$$[lm] \equiv \text{sine of the angle } \widehat{lm}$$

$$[lm/pq] \equiv \begin{Vmatrix} l_1 & l_2 & l_3 & l_4 \\ m_1 & m_2 & m_3 & m_4 \end{Vmatrix} \times \begin{Vmatrix} p_1 & p_2 & p_3 & p_4 \\ q_1 & q_2 & q_3 & q_4 \end{Vmatrix}$$

$$[lmpq]^2 \equiv \begin{vmatrix} l_1 & l_2 & l_3 & l_4 \\ m_1 & m_2 & m_3 & m_4 \\ p_1 & p_2 & p_3 & p_4 \\ q_1 & q_2 & q_3 & q_4 \end{vmatrix}^2$$

$$= \begin{vmatrix} 1 & (lm) & (lp) & (lq) \\ (lm) & 1 & (mp) & (mq) \\ (lp) & (mp) & 1 & (pq) \\ (lq) & (mq) & (pq) & 1 \end{vmatrix}$$

Hence

$$\mathbf{H} \equiv [lmpq]^2 - [lm]^2[pq]^2 - [lm/pq]^2. \qquad (82.3$$

The equation (82.2) is a quadratic in $\cos^2 \theta$, and consequently, there are two values of $\cos^2 \theta$. Suppose ϕ and ψ are the two values of θ corresponding to these two values of $\cos^2 \theta$, and we obtain

$$\cos^2 \phi . \cos^2 \psi = [lm/pq]^2 / [lm]^2[pq]^2 \qquad (82.4$$

and changing all cosines into sines in (82.2), we obtain

$$\sin^2 \phi . \sin^2 \psi = [lmpq]^2 / [lm]^2[pq]^2. \qquad (82.5$$

It can be easily proved that the functions on the right in these equations are invariant under any scheme of orthogonal transformation.

Thus, these two functions may be taken to define the angular invariants referred to in § 62.

It is interesting to note that the product of the cosines and of the sines of the two angles between the planes α and β are invariant. The two angles then appear in very convenient forms under the product of their sines or cosines.

We may define the function $\cos \phi . \cos \psi$ as the "Index of projectivity," and the function $\sin \phi . \sin \psi$ as the "Index of conjectivity" for the two given planes α and β.

It may be further shown that the two roots of the equation (82.2) are real, and consequently the two angles ϕ and ψ are also real.

That the angles ϕ and ψ are invariant appears from the fact that $\phi + \psi$ and $\phi - \psi$ are invariant.

If the two planes are specified each by a pair of linear equations, it will be seen that the analysis follows the same course as above, and the above results are obtained without any difficulty, the reason for which will appear in the sequel.

It is easily seen that, if either of ϕ or ψ vanishes, the planes α and β intersect in a line, and are, therefore, situated in the same hyperplane, the other now becomes the dihedral angle between α and β. When both ϕ and ψ vanish, the two planes coincide or are completely parallel. Again, when $\phi=\pi/2$, $\psi\neq0$, the two planes are called *half-orthogonal*. When $\phi=\pi/2$, $\psi=0$, the two planes lie in the same hyperplane and are called *simply perpendicular*. When $\phi=\pi/2$, $\psi=\pi/2$, the planes are said to be *absolutely* or *completely perpendicular*. These results are also derivable from analytical considerations, as will be demonstrated subsequently.

83. Particular Cases:

The two angles ϕ and ψ are independent and may admit of all possible values. If both are zero, the two planes α and β coincide or are *completely parallel* or are *half-parallel*. If one of the angles ϕ or ψ be zero, while the other is not zero, the planes intersect in a line. If one of the two angles is a right angle, the other being neither zero nor a right angle, the two planes are said to be *half-orthogonal*. When one is a right angle, while the other is zero, the planes are said to be *simply perpendicular*. When both are right angles, the two planes are *absolutely* or *completely orthogonal*. Thus, in the Fourfold, there are three distinct varieties of orthogonal planes.

84. Isocline Planes :

The case when $\phi=\psi$, *i.e.*, when the two critical angles are equal, deserves special consideration. In this case, the four lines a_∞, b_∞, a^*_∞, b^*_∞ have an infinite number of transversals, which are, in fact, the generators of the opposite system of the ruled surface generated by the four lines. These transversals with O will determine an infinite number of planes, all of which intersect the two given planes orthogonally. Hence, there is an infinite number of planes on which the two given planes cut out equal angles and these latter are said to be "isocline." In this case there is neither a maximum nor a minimum angle. Considering the pencils of lines in the planes, it is seen that the two superimposed homographic pencils (§ 75) have more than two double rays. The theory of isocline planes through a point is analogous to the theory of equidistant straight lines in Elliptic Geometry of three dimensions, which are called Clifford's parallels or *paratactic* lines. It follows, then, that if α and β are isocline, so also are any two planes which cut them both orthogonally.

85. The Case of one critical Angle zero :

When one angle ϕ is zero, the two planes have a common line, *i.e.*, they lie in the same hyperplane. In this case, the other angle is still such that each of its arms is the projection of the other. The zero angle may be regarded as being formed by the line of intersection (taken twice), which lies in both the planes. The plane of this zero angle is then a plane through this common line orthogonal to both the planes. It is to be

noted, however, that this plane does not lie in the same hyperplane with α and β.

In this case the lines at infinity a_∞, b_∞ have a common point P_∞ on the line of intersection of α and β, and the lines a_∞^*, b_∞^*, normally associated to them, have also a common point of intersection (say L_∞); for, the pencil of parallel rays, which are orthogonal to the hyperplane of α and β, passes through a common point L_∞ at infinity, normally associated with a_∞ and b_∞, and is therefore the point of intersection of a_∞^* and b_∞^*. It is evident then that the common transversals of the four lines a_∞, b_∞, a_∞^*, b_∞^* are the line $P_\infty L_\infty$ and the line of intersection of the planes (a_∞, b_∞) and (a_∞^*, b_∞^*). The line $P_\infty L_\infty$ determines with O a plane through the common line of α and β, but does not lie in the same hyperplane, and this is the plane of the zero angle, and is orthogonal to both α and β. The line of intersection of the planes (a_∞, b_∞), (a_∞^*, b_∞^*) determines with O a plane, which lies in the hyperplane of α and β, since it has two lines common with the hyperplane, namely, the arms of the non-zero angle ψ between α and β. This angle ψ is the ordinary dihedral angle between the two planes, which has been called a *relative maximum*, *i.e.*, the greatest angle which a line of one plane can make with its projection on the other plane.

86. Half-orthogonal Planes:

When α and β are half-orthogonal, the line at infinity a_∞ of α must intersect the line b_∞^*, which is normally associated with the line at infinity b_∞ of β, and b_∞ intersects a_∞^*, which is normally associated with a_∞. Hence,

the four lines a_∞, b_∞, a_∞^*, b_∞^* have two points common. The two transversals are the lines joining these two points and the line of intersection of the two planes in which they lie. The first gives at O an angle $90°$, and this is the angle ψ, the other line gives a non-zero angle ϕ. Hence, for two half-orthogonal planes, one angle is ϕ and the other is the maximum angle $90°$. If $\phi=0$, and $\psi\neq 0$, the two planes intersect in a line. If, however, $\psi=90°$, the two planes are orthogonal in an ordinary hyperplane. When $\phi=\psi=90°$, α and β are absolutely orthogonal, and $a_\infty \equiv a_\infty^*$, $b_\infty \equiv b_\infty^*$.

87. Angular Invariants of two Planes :

In § 82, we have deduced the formulae

$$\cos^2\phi.\cos^2\psi=[lm/pq]^2/[lm]^2.[pq]^2$$

$$\sin^2\phi.\sin^2\psi=[\,lmpq\,]^2/[lm]^2[pq]^2.$$

We shall now prove that the functions on the right are invariantive for all variations in the choice of the guiding lines of the planes.

Choose, as the guiding lines of the first plane (l, m) any two non-coincident directions (l'_i, m'_i) in the plane (l, m) :—

$$l'_1=\alpha l_1+\beta m_1, \qquad l'_2=\alpha l_2+\beta m_2,$$

$$l'_3=\alpha l_3+\beta m_3, \qquad l'_4=\alpha l_4+\beta m_4$$

$$m'_1=\gamma l_1+\delta m_1, \qquad m'_2=\gamma l_2+\delta m_2,$$

$$m'_3=\gamma l_3+\delta m_3, \qquad m'_4=\gamma l_4+\delta m_4$$

where $$\alpha\delta-\beta\gamma\neq 0.$$

Similarly, choose two non-coincident lines p'_i, q'_i in the plane (p, q) :—

$$p'_1=\alpha' p_1+\beta' q_1, \qquad p'_2=\alpha' p_2+\beta' q_2,$$

$$p'_3=\alpha' p_3+\beta' q_3, \qquad p'_4=\alpha' p_4+\beta' q_4$$

$$q'_1=\gamma' p_1+\delta' q_1, \qquad q'_2=\gamma' p_2+\delta' q_2,$$

$$q'_3=\gamma' p_3+\delta' q_3, \qquad q'_4=\gamma' p_4+\delta' q_4$$

$$\text{where} \qquad \alpha'\delta'-\beta'\gamma' \neq 0.$$

Now,

$$[l'm'/p'q']=\Sigma\begin{vmatrix} l'_1 & m'_1 \\ l'_2 & m'_2 \end{vmatrix}\times\begin{vmatrix} p'_1 & q'_1 \\ p'_2 & q'_2 \end{vmatrix}$$

$$=\Sigma\begin{vmatrix} \alpha l_1+\beta m_1 & \gamma l_1+\delta m_1 \\ \alpha l_2+\beta m_2 & \gamma l_2+\delta m_2 \end{vmatrix}\times\begin{vmatrix} \alpha' p_1+\beta' q_1 & \gamma' p_1+\delta' q_1 \\ \alpha' p_2+\beta' q_2 & \gamma' p_2+\delta' q_2 \end{vmatrix}$$

$$=(\alpha\delta-\beta\gamma)(\alpha'\delta'-\beta'\gamma')\Sigma\begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix}\times\begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix}$$

$$=(\alpha\delta-\beta\gamma)(\alpha'\delta'-\beta'\gamma')[lm/pq].$$

Similarly,

$$[l'm']^2=\Sigma\begin{vmatrix} l'_1 & m'_1 \\ l'_2 & m'_2 \end{vmatrix}^2=(\alpha\delta-\beta\gamma)^2\Sigma\begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix}^2$$

$$=(\alpha\delta-\beta\gamma)^2[lm]^2,$$

so that $\qquad [l'm']=(\alpha\delta-\beta\gamma)[lm]$

and $\qquad [p'q']=(\alpha'\delta'-\beta'\gamma')[pq]$

$$\therefore \frac{[l'm'/p'q']}{[l'm'][p'q']}=\frac{(\alpha\delta-\beta\gamma)(\alpha'\delta'-\beta'\gamma')}{(\alpha\delta-\beta\gamma)(\alpha'\delta'-\beta'\gamma')}\cdot\frac{[lm/pq]}{[lm][pq]}$$

$$=\frac{[lm/pq]}{[lm][pq]}. \tag{87.1}$$

In an analogous manner, it can be shown that the function $[lmpq]/[lm][pq]$ is an invariant for the two planes. These are then the two angular invariants and may be taken, as they actually have been, to define the inclination of the two planes.

If, then, an angle χ can be found such that

$$\cos \chi = \cos \phi . \cos \psi,$$

then χ may be called the "virtual angle" between the planes.

If ϕ and ψ are two sides of a spherical right-angled triangle in the ordinary space, then the side opposite to the right angle is given by the formula

$$\cos \chi = \cos \phi . \cos \psi . \tag{87.2}$$

Such a real angle χ can, therefore, be determined, but cannot be organically specified, having a unique location in reference to the planes.*

88. Significance of Orientation-Cosines of a Plane:

From what is stated above, we are now in a position to realise the significance and justification for the use of the term 'orientation-cosines' of a plane explained in § 33.

If l_i and m_i $(i=1, 2, 3, 4)$ are the guiding lines of a plane, including an angle ω between them, and if

* A. R. Forsyth calculates the first function and calls it $\cos \phi$, where ϕ is the inclination of the two planes, but does not indicate the nature of ϕ, which, according to his analysis, seems to be an ordinary plane-angle, but later on he finds a contradiction, which is nowhere explained. —*Geometry of Four Dimensions*, Vol. I (1930)—§§ 83 and 92.

ϕ_{12} and ψ_{12} are the critical angles which this plane makes with the co-ordinate plane (x_1, x_2),* then

$$\cos \phi_{12}.\cos \psi_{12}. \sin \omega = l_1 m_2 - l_2 m_1.$$

If, then, $\triangle$ is the area of a triangle in the plane and $\triangle_{12}$ its projection on the co-ordinate plane of (x_1, x_2), we have—

$$\triangle \cos \phi_{12}.\cos \psi_{12} = \triangle.\frac{l_1 m_2 - l_2 m_1}{\sin \omega} = \triangle_{12}$$

i.e., $$\cos \phi_{12}.\cos \psi_{12} = \triangle_{12}/\triangle. \qquad (88.1$$

Hence, it follows that the orientation-cosines of a plane are the *indices of projectivity* between the given plane and the six co-ordinate planes ; and each of them is the product of the cosines of the critical angles between the plane and the corresponding co-ordinate plane.

Thus, the quantities a, b, c, f, g, h, or more appropriately the symbols with subscripts, namely, L_{12}, L_{23}, L_{34}, L_{31}, L_{14}, L_{12}, (§ 33) are multipliers which determine the projections of any plane area upon the co-ordinate planes.

It appears, therefore, that the orientation-cosines determine the orientation of a plane in reference to the co-ordinate planes, just as the direction-cosines of a line determine its direction in reference to the co-ordinate axes; but it must be clearly understood that these orientation-cosines in no way specify the location of the plane. This is because within the plane there is no unique line, nor at any point thereof there is any unique direction perpendicular to the plane. Hence,

* From this stage we shall speak of the co-ordinate axes as the axes of x_1, x_2, x_3, x_4, unless otherwise expressly mentioned, as this notation will be very helpful in studying the inclination of planes and their orientation-cosines.

the orientation of a plane is made more definite by the use of co-ordinate planes of reference and not the co-ordinate axes.

From what has been explained in § 33, it is now easily deduced that the orientation-cosines of a plane absolutely perpendicular to the plane (a, b, c, f, g, h) are (f, g, h, a, b, c).

89. Three Varieties of orthogonal Planes:

The angular invariants for two planes may be written in the forms—

$$\cos \chi = \cos \phi . \cos \psi = [lm/pq]/[lm][pq] \qquad \dots \quad (1)$$

$$\sin \chi = \sin \phi . \sin \psi = [lmpq]/[lm][pq] \qquad \dots \quad (2)$$

A necessary condition that the two planes may be at right angles is that

$$\chi = \tfrac{1}{2}\pi, \text{ or, } \cos \chi = \cos \phi . \cos \psi = 0 \qquad \dots \quad (89.1$$

and consequently,

$$(lp)(mq) - (lq)(mp) = 0 \qquad \dots \quad (89.2$$

Also,

$$\sin \chi = 1, \ \textit{i.e.}, \ [lmpq] = [lm][pq] \qquad \dots \quad (89.3$$

These conditions may be fulfilled under three different circumstances, and in each case, therefore, there is a type of relation placing the two planes at right angles to each other. In the first case, condition (1) will be satisfied, if one of the critical angles (say ϕ) is $\frac{1}{2}\pi$, while the other $\psi = 0$. In this case the two planes will have a common line of section, and consequently, they will lie in the same hyperplane. In this case the two planes are said to be " simply perpendicular," and

there is one direction in one plane which is perpendicular to every direction in the other. In the second case, one of the critical angles (say ϕ) is $\frac{1}{2}\pi$, while the other $\psi \neq 0$. The planes are then said to be "half-orthogonal," as it will be seen that in this case there is nothing distinctive to any line assumed arbitrarily in one plane; there can always be chosen a line in the other which is perpendicular to the former.

In the third case, the condition (89.2) will be satisfied, when both the critical angles ϕ and ψ are right angles. In this case, every line in one plane is at right angles to every line in the other. This is a case peculiar to the Fourfold, and perpendicularity is obtained in its complete form. The planes are said to be "absolutely or completely perpendicular."

The three distinctive cases must then fulfil the following conditions:—

(1) $[lm/pq]=0$ and $[lmpq]=0$ (*simply perpendicular*)

(2) $[lm/pq]=0$ and $[lmpq]\neq 0$,

or, $[lmpq]\neq[lm][pq]$ (*half-orthogonal*)

(3) $[lm/pq]=0$, $[lmpq]=[lm][pq]$ (*absolutely orthogonal*)

These notational conditions, when expressed in full length, may be written in the following forms:

$$[lm/pq]=\cos\theta_{13}.\cos\theta_{24}-\cos\theta_{14}.\cos\theta_{23}=0 \qquad (89.4)$$

$$[lm][pq]=\sin\theta_{12}.\sin\theta_{34}$$

where $\theta_{12}=\widehat{lm},\quad \theta_{34}=\widehat{pq},\quad \theta_{13}=\widehat{lp},$

$\theta_{14}=\widehat{lq},\quad \theta_{23}=\widehat{mp},\quad \theta_{24}=\widehat{mq}$

$$[lmpq]^2 \equiv \begin{vmatrix} l_1 & l_2 & l_3 & l_4 \\ m_1 & m_2 & m_3 & m_4 \\ p_1 & p_2 & p_3 & p_4 \\ q_1 & q_2 & q_3 & q_4 \end{vmatrix}^2 = \begin{vmatrix} \Sigma l^2 & \Sigma lm & \Sigma lp & \Sigma lq \\ \Sigma lm & \Sigma m^2 & \Sigma mp & \Sigma mq \\ \Sigma lp & \Sigma mp & \Sigma p^2 & \Sigma pq \\ \Sigma lq & \Sigma mq & \Sigma pq & \Sigma q^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \cos\theta_{12} & \cos\theta_{13} & \cos\theta_{14} \\ \cos\theta_{12} & 1 & \cos\theta_{23} & \cos\theta_{24} \\ \cos\theta_{13} & \cos\theta_{23} & 1 & \cos\theta_{34} \\ \cos\theta_{14} & \cos\theta_{24} & \cos\theta_{34} & 1 \end{vmatrix} \equiv \Theta^2 (\text{say}) = 0. \qquad (89.5$$

Hence, a necessary condition of orthogonality in all cases is—

$$[lm/pq] \equiv \cos\theta_{13}.\cos\theta_{24} - \cos\theta_{14}.\cos\theta_{23} = 0 \qquad (89.6$$

In the case of simply orthogonal planes, an additional condition, namely, $\Theta = 0$ must be satified, and this implies that the planes lie in the same hyperplane.

In the case of half orthogonal planes,

$$\Theta \neq o, \text{ or, } \Theta \neq \sin\theta_{12}.\sin\theta_{34} \qquad \dots \quad (89.7$$

In the case of absolutely perpendicular planes, a further condition, namely,

$$[lmpq] = [lm][pq], \ i.e., \ \Theta = \sin\theta_{12}.\ \sin\theta_{34} \qquad (89.8$$

must be satisfied for all values of θ_{12}, θ_{34}, as these are the angles between the guiding lines of the planes. This requires that

$$\cos\theta_{13} = 0,\ \cos\theta_{24} = 0,\ \cos\theta_{14} = 0,\ \cos\theta_{23} = 0. \qquad (89.9$$

If these latter conditions are assured, the preceding condition is also satisfied, and it follows from what has

been stated before, that any line in the first plane is perpendicular to all lines in the second, and *vice versa*. Hence, the conditions both necessary and sufficient that any two planes may be absolutely perpendicular are that the four angles $\theta_{13}, \theta_{14}, \theta_{23}, \theta_{24}$ are right angles, *i.e.*, any two guiding lines in one are perpendicular to any two lines in the other.

90. Orthogonal Frame :

In the case of two absolutely orthogonal planes, we may always choose the pairs of guiding lines mutually at right angles, *i.e.*, we may choose $\theta_{12}=\theta_{34}=\frac{1}{2}\pi$.

Hence, without affecting the orientation of the planes, we may always choose the four guiding lines, mutually orthogonal, so as to form an orthogonal frame of axes, and the two given planes as two of the co-ordinate planes. Four remaining planes, two pairs again mutually absolutely orthogonal, are thereby determined. Any one of these four lines is at right angles to the remaining three and is consequently normal to the hyperplane determined by them. Hence, in any orthogonal frame, there are four mutually orthogonal hyperplanes of reference, six mutually orthogonal planes of reference and four mutually orthogonal axes of reference.

91. Inclination of two Planes in terms of their Orientation-Cosines :

Let (a, b, c, f, g, h) and (a', b', c', f', g', h') respectively be the orientation-cosines of any two planes drawn through the origin. Let $\triangle$ denote the

area of a triangle in the first plane, whose projection on the second plane is denoted by $\triangle'$. If, then, ϕ and ψ be the critical angles between the planes, it follows from what has been shown in § 88 that the index of projectivity between the planes is given by

$$\cos\phi.\cos\psi=\frac{\triangle'}{\triangle}.$$

If l_i and m_i are the guiding lines of the first plane and l_i', m_i' those of the second $(i=1, 2, 3, 4)$ we have (§ 82)

$\sin\omega.\sin\omega'\cos\phi.\cos\psi=$

$$\left\|\begin{matrix} l_1 & l_2 & l_3 & l_4 \\ m_1 & m_2 & m_3 & m_4 \end{matrix}\right\| \times \left\|\begin{matrix} l_1' & l_2' & l_3' & l_4' \\ m_1' & m_2' & m_3' & m_4' \end{matrix}\right\|$$

$$=\Sigma\begin{vmatrix} l_1 & l_2 \\ m_1 & m_2 \end{vmatrix}\times\begin{vmatrix} l_1' & l_2' \\ m_1' & m_2' \end{vmatrix},$$

where $\omega=\widehat{lm}$, and $\omega'=\widehat{l'm'}$.

$$\therefore\quad \cos\phi.\cos\psi=\frac{\Sigma L_{12}.L'_{12}}{\sin\omega\ \sin\omega'}=\Sigma\frac{L_{12}}{\sin\omega}.\frac{L'_{12}}{\sin\omega'}=\Sigma aa'.$$

Thus,

$$\cos\phi.\cos\psi=\frac{\triangle'}{\triangle}=aa'+bb'+cc'+ff'+gg'+hh'. \qquad (91.1$$

If, then, we project the area $\triangle$ on the co-ordinate planes, the projections are $a\triangle$, $b\triangle$, $c\triangle$, $f\triangle$, $g\triangle$, $h\triangle$. If these are again projected on the second plane, we respectively obtain

$$a'.a\triangle,\ b'.b\triangle,\ c'.c\triangle,\ f'.f\triangle,\ g'.g\triangle,\ h'.h\triangle$$

and the sum of these areas

$$=(aa'+bb'+cc'+ff'+gg'+hh')\,\triangle=\triangle'. \qquad (91.2$$

Thus, we obtain the theorem that *the projection of a plane area upon any other plane is equal to the sum of the projections upon the same plane of its projections on the co-ordinate planes.*

It is to be noted then that in the fourfold plane-areas can be projected upon co-ordinate planes and these projections again projected upon a second plane and combined, just as lines in the ordinary space are projected on the co-ordinate axes and these projected again on any line and combined to obtain its projection on the same.

92. Conditions of Orthogonality in terms of Orientation-Cosines :

The two planes are evidently orthogonal, if

$$aa'+bb'+cc'+ff'+gg'+hh'=0. \qquad \ldots \quad (92.1$$

The two planes will lie in the same hyperplane, if $\Theta=0$, *i.e.*, if

$$\begin{vmatrix} l_1 & l_2 & l_3 & l_4 \\ m_1 & m_2 & m_3 & m_4 \\ l_1' & l_2' & l_3' & l_4' \\ m_1' & m_2' & m_3' & m_4' \end{vmatrix}=0.$$

This determinant equation, when expanded in terms of the minors of the first two rows, can be written in the form

$$af'+bg'+ch'+a'f+b'g+c'h=0. \qquad (92.2$$

Hence, the condition (92.1) implies that the plane (a, b, c, f, g, h) lies in the same hyperplane with a plane whose orientation-cosines are respectively f', g', h', a', b', c', *i.e.*, the plane absolutely perpendicular to the plane (a', b', c', f', g', h').

Therefore, when both these conditions are satisfied, the two planes lie in the same hyperplane and are perpendicular to each other, *i.e.*, they are simply perpendicular. It is also seen that in this case the first plane intersects the plane absolutely perpendicular to the second. Hence we conclude that *if a plane intersects another plane and its absolutely perpendicular plane, it is simply perpendicular to the latter, as well as to the former.*

The condition (92.2) implies that the plane (a, b, c, f, g, h) is perpendicular to the plane (f', g', h', a', b', c'), *i.e.*, to the plane absolutely orthogonal to the plane (a', b', c', f', g', h').

Hence, *a plane is perpendicular to another, when it intersects the absolutely orthogonal plane to the latter.*

If the two planes are mutually absolutely perpendicular, we have

$$\frac{a}{f'}=\frac{b}{g'}=\frac{c}{h'}=\frac{f}{a'}=\frac{g}{b'}=\frac{h}{c'}. \qquad \ldots \quad (92.3$$

Thus, the conditions (92.1) and (92.2) hold for two simply perpendicular planes and condition (92.3) holds for absolutely perpendicular planes. Condition (92.1) is necessary for half-orthogonal planes.

If the plane $\alpha(a, b, c, f, g, h)$ intersects the plane $\beta(a', b', c', f', g', h')$ and its absolutely perpendicular

plane $\beta'(f', g', h', a', b', c')$ in lines, conditions (92.1) and (92.2) are satisfied, and it follows that α is perpendicular to both β and β'. Hence, *a plane intersecting in a line each of two absolutely perpendicular planes is perpendicular to both.*

Cor. : The critical planes of two given planes are common perpendicular between them and intersect them both. They also intersect their absolutely perpendicular planes. Hence, the common perpendicular planes of any two planes intersect all the four planes, namely, the two given planes and their absolutely perpendicular planes, as is otherwise shown in § 81.

93. Isocline Planes :

The case when the two values of θ obtained in § 82 are equal in magnitude deserves special consideration. If, then, the two values of θ being ϕ and ψ,

$$\psi = \pm\phi,$$

the angular invariants (82.4) and (82.5) take the forms

$$\cos^2\psi = [lm/pq]/[lm][pq]$$

$$\sin^2\psi = [lmpq]/[lm][pq]$$

so that

$$\frac{[lm/pq]+[lmpq]}{[lm][pq]} = \cos^2\psi + \sin^2\psi = 1.$$

In this case, the two planes are said to be "isocline." Hence, the condition, necessary for the two planes being isocline, is

$$[lm/pq]+[lmpq] = [lm][pq] \quad \dots \quad (93.1$$

or, $\cos\theta_{13}.\cos\theta_{24} - \cos\theta_{14}\cos\theta_{23} + \Theta^{\frac{1}{2}} = \sin\theta_{12}.\sin\theta_{34}$. (93.2

If we put

$$[lm][pq] \equiv a, \quad [lmpq] \equiv b, \quad [lm/pq] \equiv c,$$

then the equation (82.2) may be written in the form

$$a^2\cos^4\theta + (b^2 - a^2 - c^2)\cos^2\theta + c^2 = 0$$

whence

$$\cos^2\theta = \frac{(a^2+c^2-b^2) \pm \sqrt{(b^2-a^2-c^2)^2 - 4a^2c^2}}{2a^2}$$

$$= \frac{(a^2+c^2-b^2) \pm \sqrt{a^4+b^4+c^4-2a^2b^2-2b^2c^2-2c^2a^2}}{2a^2}$$

$$= \frac{(a^2+c^2-b^2) \pm \sqrt{-(a+b+c)(b+c-a)(c+a-b)(a+b-c)}}{2a^2}.$$

$\therefore$ The roots of the equation (82.2) will be equal, if the expression under the radical sign vanishes, *i.e.*, if any one of the factors of the expression under the radical sign vanishes, *i.e.*, if

$$\vartheta \equiv \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 \\ 1 & c^2 & 0 & a^2 \\ 1 & b^2 & a^2 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & [lm/pq]^2 & [lmpq]^2 \\ 1 & [lm/pq]^2 & 0 & [lm]^2[pq]^2 \\ 1 & [lmpq]^2 & [lm]^2[pq]^2 & 0 \end{vmatrix} = 0. \qquad (93.2$$

When this condition is satisfied, the two roots are equal, and we have—

$$\cos\psi = \cos\phi \quad \therefore \quad \psi = \phi, \text{ or } -\phi.$$

The planes are said to be *isocline*, and there are two ways in which a plane may be isocline to another, according as $\psi=\phi$, or $-\phi$. These two cases must be distinguished, and we may say that in the first case the planes are 'positively isocline,' and in the second case, they are "negatively isocline."

It follows then that *there are two senses in which two planes may be isocline*.

When the planes (a, b, c, f, g, h) and (a', b', c', f', g', h') are isocline, positively or negatively, we have

$$\cos\phi.\cos\psi=\cos^2\phi=aa'+bb'+cc'+ff'+gg'+hh',$$

showing that $\Sigma\, aa'$ is a perfect square, and then we must have

$$\cos\phi=\sqrt{\Sigma aa'}\,,\ \text{or},\ \phi=\cos^{-1}\sqrt{\Sigma aa'}\,. \qquad (93.4$$

Hence, we conclude that *the condition that any two planes may be isocline is that the sum of the products of their corresponding orientation-cosines is a perfect square*. Symbolically, $\Sigma aa'$ must be a perfect square, subject to the conditions $\Sigma a^2=1$, $\Sigma a'^2=1$

$$af+bg+ch=0, \quad a'f'+b'g'+c'h'=0.$$

94. The critical Lines of two Planes :

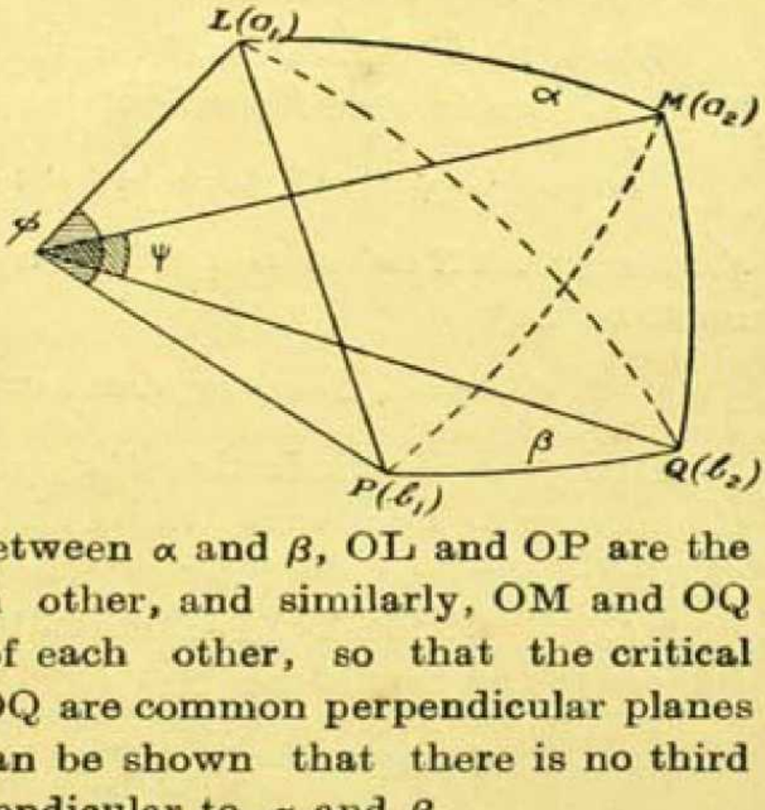
The arms of the two critical angles between two planes may be called their *minimal lines*. The two arms in each critical position may be called the *corresponding lines*, and the planes of the two angles may be called the *critical* or *minimal* planes of the two given planes. On these two planes are cut out the critical angles by the two given planes.

It is interesting to note that these two critical

planes are mutually absolutely orthogonal and each of them cuts the two given planes orthogonally.

That there are only two planes which are common perpendicular to the two given planes have already been otherwise established.

We shall now prove that *these two common perpendicular planes (critical planes) are themselves mutually absolutely orthogonal.*



Let OL and OM be the critical lines in the plane $\alpha(l, m)$, and OP and OQ those in the plane $\beta\ (p, q)$, so that $\angle \mathrm{LOP}=\phi$ and $\angle \mathrm{MOQ}=\psi$. Since ϕ and ψ are the minimum angles between α and β, OL and OP are the projections of each other, and similarly, OM and OQ are the projections of each other, so that the critical planes LOP and MOQ are common perpendicular planes of α and β, and it can be shown that there is no third plane common perpendicular to α and β.

Now, in α, the critical line OM is perpendicular to the critical line OL ; similarly, in β, the critical lines OP and OQ are mutually orthogonal.

For, the direction-cosines of the lines OL and OM are given by $\mathrm{A}l_i+\mathrm{B}m_i$ $(i=1, 2, 3, 4)$, where A and B are given by the equations (8) of § 82.

Let A_1, B_1 and A_2, B_2 be respectively the parameters of the lines OL and OM, so that OL is the line $\mathrm{A}_1l_i+\mathrm{B}_1m_i$ and OM is the line $\mathrm{A}_2l_i+\mathrm{B}_2m_i$.

Then, $\cos \text{LOM} = \Sigma(A_1 l_i + B_1 m_i)(A_2 l_i + B_2 m_i)$

$$= A_1 A_2 + (A_1 B_2 + A_2 B_1)(lm) + B_1 B_2$$

$$= B_1 B_2 \left\{ \frac{A_1 A_2}{B_1 B_2} + \left(\frac{A_1}{B_1} + \frac{A_2}{B_2} \right)(lm) + 1) \right\}. \quad \ldots \quad (1)$$

Eliminating C and D from equations (8) of § 82, we have

$$\frac{A + B(lm)}{A(lm) + B} = \frac{\lambda.A + \nu.B}{\nu.A + \mu.B}$$

where $\lambda = (lp)^2 + (lq)^2 - 2(pq)(lp)(lq)$

$$\mu = (mp)^2 + (mq)^2 - 2(pq)(mp)(mq)$$

$$\nu = (lp)(mp) + (lq)(mq) - (lq)(mp)(pq) - (pq)(lp)(mq)$$

or, $A^2\{\nu - \lambda(lm)\} + AB\{\mu - \lambda\} + B^2\{\mu(lm) - \nu\} = 0.$

This is a quadratic in $\frac{A}{B}$, giving two values, namely,

$$\frac{A_1}{B_1} \text{ and } \frac{A_2}{B_2}.$$

so that, $\frac{A_1 A_2}{B_1 B_2} = \frac{\mu(lm) - \nu}{\nu - \lambda(lm)}$

and $\frac{A_1}{B_1} + \frac{A_2}{B_2} = \frac{\lambda - \mu}{\nu - \lambda(lm)}$

$$\therefore \quad \cos \text{LOM} = B_1 B_2 \left\{ \frac{\mu(lm) - \nu}{\nu - \lambda(lm)} + \left(\frac{\lambda - \mu}{\nu - \lambda(lm)} \right)(lm) + 1 \right\}$$

$$= \frac{B_1 B_2}{\nu - \lambda(lm)} \left\{ \mu(lm) - \nu + (\lambda - \mu)(lm) + \nu - \lambda(lm) \right\} = 0$$

$\therefore$ The angle LOM$=2n\pi\pm\frac{\pi}{2}$, *i.e.*, the two critical lines OL and OM are mutually perpendicular.

Similarly, it can be shown that OP and OQ are mutually perpendicular.

Now, since OM is perpendicular to OL and lies in the plane α, which is perpendicular to the critical plane LOP, it is perpendicular to all lines lying in the same plane, and is consequently normal to the plane LOP. Similarly, the other critical line OQ is normal to the plane LOP. Thus, the critical plane MOQ, which is determined by the critical lines OM and OQ, is absolutely perpendicular to the critical plane LOP, determined by the critical lines OL and OP.

Hence, *the planes of the two critical angles between two given planes are mutually absolutely perpendicular.*

95. Theorem: *The critical angles between any two planes are equal to the critical angles between their absolutely orthogonal planes.*

Let the two given planes α and β be drawn through the origin and be defined by the guiding lines l_i, m_i and p_i, q_i respectively ($i=1, 2, 3, 4$).

Let λ_i and μ_i be the direction-cosines of two lines respectively lying in the planes α' and β' absolutely perpendicular to α and β.

If now θ denote the angle between the lines λ and μ, we have

$$\left.\begin{aligned}\cos\theta&=\lambda_1\mu_1+\lambda_2\mu_2+\lambda_3\mu_3+\lambda_4\mu_4\\ \Sigma\lambda_i^2=1 &\quad\text{and}\quad \Sigma\mu_i^2=1.\end{aligned}\right\}\quad\text{(A)}$$

Again, since λ_i lies in the plane absolutely orthogonal to the plane (l, m), it is perpendicular to all lines in the latter plane, and consequently,

$$\Sigma\lambda_i l_i = 0, \qquad \Sigma\lambda_i m_i = 0.$$

For a similar reason, (B)

$$\Sigma\mu_i p_i = 0, \qquad \Sigma\mu_i q_i = 0.$$

Differentiating equations (A) and (B), we get

$$\Sigma\lambda_i\delta\mu_i + \Sigma\mu_i\delta\lambda_i = 0, \quad \Sigma\lambda_i\delta\lambda_i = 0, \quad \Sigma\mu_i\delta\mu_i = 0,$$

$$\Sigma l_i\delta\lambda_i = 0, \quad \Sigma m_i\delta\lambda_i = 0, \quad \Sigma p_i\delta\mu_i = 0, \quad \Sigma q_i\delta\mu_i = 0.$$

Hence, multipliers A, B, C, D can be found satisfying the relations

$$A\lambda_i + B\mu_i + Cp_i + Dq_i = 0$$

$$(i = 1, 2, 3, 4). \qquad \dots \quad (1)$$

Similarly, multipliers A′, B′, C′, D′ can be found satisfying the relations

$$A'\lambda_i + B'\mu_i + C'l_i + D'm_i = 0$$

$$(i = 1, 2, 3, 4). \qquad \dots \quad (2)$$

Now, multiplying the relations (1) respectively by μ_1, μ_2, μ_3, μ_4 in order and adding, we get, in virtue of relations (A)

$$A\cos\theta + B = 0. \qquad \dots \quad (3)$$

Similarly, from (2), by multiplying by $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ respectively in order and adding we get

$$A' + B'\cos\theta = 0. \qquad \dots \quad (4)$$

Substituting these values in (1) and (2), we obtain relations of the forms

$$\lambda_i - \mu_i \cos\theta = \mathrm{P}p_i + \mathrm{Q}q_i \qquad \dots \quad (5)$$

and

$$\mu_i - \lambda_i \cos\theta = \mathrm{P}'l_i + \mathrm{Q}'m_i \qquad \dots \quad (6)$$

$$(i = 1, 2, 3, 4)$$

where P, Q, P′, Q′ are new parameters.

Now, multiplying the relations (5) respectively by l_1, l_2, l_3, l_4 in order and adding, we obtain

$$-(\mu l)\cos\theta = \mathrm{P}(lp) + \mathrm{Q}(lq). \qquad \dots \quad (7)$$

Similarly, multiplying the same relations (5) respectively by m_1, m_2, m_3, m_4 in order and adding, we get

$$-(\mu m)\cos\theta = \mathrm{P}(mp) + \mathrm{Q}(mq). \qquad \dots \quad (8)$$

Proceeding exactly in the same manner with relations (6), we obtain

$$(\mu l) = \mathrm{P}' + \mathrm{Q}'(lm) \qquad \dots \quad (9)$$

$$(\mu m) = \mathrm{P}'(lm) + \mathrm{Q}'. \qquad \dots \quad (10)$$

From these four relations (7), (8), (9), (10) by eliminating (μl) and (μm), we obtain

$$\{\mathrm{P}' + \mathrm{Q}'(lm)\}\cos\theta + \mathrm{P}(lp) + \mathrm{Q}(lq) = 0 \qquad \dots \quad (11)$$

$$\{\mathrm{P}'(lm) + \mathrm{Q}'\}\cos\theta + \mathrm{P}(mp) + \mathrm{Q}(mq) = 0. \qquad \dots \quad (12)$$

Dealing with relations (5) and (6) in the same manner, we again obtain

$$\{\mathrm{P} + \mathrm{Q}(pq)\}\cos\theta + \mathrm{P}'(lp) + \mathrm{Q}'(mp) = 0 \qquad \dots \quad (13)$$

$$\{\mathrm{P}(pq) + \mathrm{Q}\}\cos\theta + \mathrm{P}'(lq) + \mathrm{Q}'(mq) = 0. \qquad \dots \quad (14)$$

Finally, eliminating P, Q, P′, Q′ from relations (11)—(14), we obtain the following determinant equation for determining θ :—

$$\begin{vmatrix} \cos\theta & (lm)\cos\theta & (lp) & (lq) \\ (lm)\cos\theta & \cos\theta & (mp) & (mq) \\ (lp) & (mp) & \cos\theta & \cos\theta\,(pq) \\ (lq) & (mq) & (pq)\cos\theta & \cos\theta \end{vmatrix} = 0. \qquad (95.1$$

This is the same equation as was obtained in § 82 and therefore gives the same values of θ as for the critical angles between the planes α and β. Hence, the critical angles between any two planes are equal to the critical angles between their absolutely perpendicular planes.

Cor. 1: Since the orientation-cosines of a plane are the indices of projectivity between the plane and the co-ordinate planes, it follows that they are respectively equal to the indices of projectivity between its absolutely perpendicular plane and the corresponding absolutely perpendicular co-ordinate plane. Thus, if the plane α have orientation-cosines $a_{12}, b_{23}, c_{31}, f_{34}, g_{14}, h_{24}$, the suffixes denoting the corresponding co-ordinate planes, then the orientation-cosines of the absolutely perpendicular plane α' are $f_{12}, g_{23}, h_{31}, a_{34}, b_{14}, c_{24}$, *i.e.*, the orientation-cosines of the absolutely perpendicular plane to $\alpha(a, b, c, f, g, h)$ are (f, g, h, a, b, c) respectively.

Cor. 2: The inclination of the two planes are given by

$$\cos\chi = \cos\phi . \cos\psi = aa' + bb' + cc' + ff' + gg' + hh'$$

and that between their absolutely perpendicular planes is given by

$$\cos \chi = \cos \phi . \cos \psi = ff' + gg' + hh' + aa' + bb' + cc'.$$

96. Theorem: *The critical angles between any two planes are the complements of the critical angles between any of them and the plane absolutely perpendicular to the other.*

Let the planes α and β be defined as above and let λ_i be a line in the plane absolutely perpendicular to the plane α, and μ_i be a line in the plane β.

Then we have

$$\Sigma\lambda_i l_i = 0, \quad \Sigma\lambda_i m_i = 0, \quad \Sigma\lambda_i{}^2 = 1 \qquad \ldots \quad (A)$$

We may also take

$$\mu_i = Ap_i + Bq_i \ (i = 1, 2, 3, 4)$$

$$\therefore \quad \Sigma\mu_i{}^2 = 1 = A^2 + B^2 + 2AB(pq). \qquad \ldots \quad (B)$$

If ϕ be the angle between the lines λ and μ, we have

$$\cos \phi = \Sigma\lambda_i\mu_i = \Sigma\lambda_i(Ap_i + Bq_i)$$

$$= A(\lambda p) + B(\lambda q). \qquad \ldots \quad (C)$$

Differentiating relations (A), (B) and (C), we obtain

$$\Sigma l_i \delta\lambda_i = 0, \quad \Sigma m_i \delta\lambda_i = 0, \quad \Sigma\lambda_i \delta\lambda_i = 0 \quad \ldots \quad (A')$$

$$\{A + B(pq)\}\delta A + \{A(pq) + B\}\delta B = 0 \qquad \ldots \quad (B')$$

$$\Sigma\mu_i\delta\lambda_i + \Sigma\lambda_i\delta\mu_i = \Sigma(Ap_i + Bq_i)\delta\lambda_i + \Sigma\lambda_i(p_i\delta A + q_i\delta B)$$

$$= \Sigma(Ap_i + Bq_i)\delta\lambda_i + (\lambda p)\delta A + (\lambda q)\delta B = 0. \quad (C')$$

Hence, multiplier k may be found such that

$$\left.\begin{aligned} k(\lambda p) &= \mathrm{A}+\mathrm{B}(pq) \\ k(\lambda q) &= \mathrm{A}(pq)+\mathrm{B}. \end{aligned}\right\} \quad \ldots \quad (1)$$

Also, multipliers a, b, c can be found satisfying the relations

$$al_i + bm_i + c\lambda_i + (\mathrm{A}p_i + \mathrm{B}q_i) = 0$$

$$(i=1,\ 2,\ 3,\ 4). \quad \ldots \quad (2)$$

Multiplying (1) respectively by A and B, and adding, we get

$$k \cos \phi = 1.$$

Substituting this value of k in (1), we obtain

$$\left.\begin{aligned} \cos\phi\{\mathrm{A}+\mathrm{B}(pq)\} &= (\lambda p) \\ \cos\phi\{\mathrm{A}(pq)+\mathrm{B}\} &= (\lambda q). \end{aligned}\right\} \quad \ldots \quad (3)$$

Multiplying (2) successively by l_i, m_i, λ_i, p_i, q_i in order $(i=1, 2, 3, 4)$ and adding, we get

$$\left.\begin{aligned} a + b(lm) + \mathrm{A}(lp) + \mathrm{B}(lq) &= 0 \\ a(lm) + b + \mathrm{A}(mp) + \mathrm{B}(mq) &= 0 \\ c + \mathrm{A}(\lambda p) + \mathrm{B}(\lambda q) &= 0 \\ a(lp) + b(mp) + c(\lambda p) + \mathrm{A} + \mathrm{B}(pq) &= 0 \\ a(lq) + b(mq) + c(\lambda q) + \mathrm{A}(pq) + \mathrm{B} &= 0 \end{aligned}\right\} \quad \ldots \quad (4)$$

From (4) we obtain, in virtue of retations (3)

$$\begin{aligned} c &= -\mathrm{A}(\lambda p) - \mathrm{B}(\lambda q) \\ &= -\cos\phi\{\mathrm{A}^2+\mathrm{B}^2+2\mathrm{AB}(pq)\} \\ &= -\cos\phi \end{aligned}$$

and
$$c(\lambda p) = -\cos^2\phi\{A + B(pq)\}$$
$$c(\lambda q) = -\cos^2\phi\{A(pq) + B\}.$$

Eliminating c, (λp) and (λq), the relations (4) can now be written in the forms

$$\begin{aligned}
&A\sin^2\phi &&+ B\sin^2\phi(pq) &&+ a(lp) &&+ b(mp) = 0\\
&A\sin^2\phi(pq) &&+ B\sin^2\phi &&+ a(lq) &&+ b(mq) = 0\\
&A\,(lp) &&+ B(lq) &&+ a &&+ b(lm) = 0\\
&A\,(mp) &&+ B(mq) &&+ a(lm) &&+ b = 0.
\end{aligned}$$

Hence eliminating A, B, a, b between these four equations, we obtain the following determinant equation for determining the value of ϕ:—

$$\begin{vmatrix} \sin^2\phi & \sin^2\phi(pq) & (lp) & (mp) \\ \sin^2\phi(pq) & \sin^2\phi & (lq) & (mq) \\ (lp) & (lq) & 1 & (lm) \\ (mp) & (mq) & (lm) & 1 \end{vmatrix} = 0. \quad \ldots \quad (96.1$$

If we now replace ϕ by $\frac{1}{2}\pi - \theta$, $\sin\phi = \sin(\frac{1}{2}\pi - \theta) = \cos\theta$, and the determinant reduces to that obtained in § 82.

Hence, the two values of ϕ given by (96.1) are the complements of the values of θ obtained in § 82, *i.e.*, the critical angles between the planes α and β are the complements of the critical angles between β and the plane α', absolutely perpendicular to α.

If a, b, c, f, g, h and a', b', c', f', g', h' be the orientation-cosines of the two planes, their inclination is given by (§ 91)

$$\cos\theta = \cos\phi.\cos\psi = aa' + bb' + cc' + ff' + gg' + hh'.$$

But the orientation-cosines of the plane absolutely orthogonal to the first plane are f, g, h, a, b, c; and hence, its inclination χ with the second plane is given by

$$\cos \chi \equiv \sin \phi \, . \, \sin \psi = a'f + b'g + c'h + af' + bg' + ch'.$$

97. Conditions of Parallelism:

In § 43 we have seen that two planes may intersect in only one point at infinity, or they may intersect in a line at infinity. In the first case, they are said to be *half-parallel* and in the second case, *full-parallel*, or *completely parallel*.

In both these cases, the critical angles are both zero, and consequently, the necessary conditions are obtained by putting $\phi = \psi = 0$ in the expressions for angular invariants in the forms

$$[lm/pq] = [lm][pq] \tag{97.1}$$

and

$$[lmpq] = 0. \tag{97.2}$$

When the condition (97.1) is satisfied and not the condition (97.2), the planes are half-parallel; when both the conditions are satisfied, the planes are completely parallel. In fact, the second condition shows that the planes lie in the same hyperplane. It should be noted that these conditions are only necessary, but not sufficient. These along with the conditions of § 43 form both necessary and sufficient conditions for parallelism of two planes.

If the planes be defined by their orientation-cosines, then the condition (97.1) becomes

$$aa' + bb' + cc' + ff' + gg' + hh' = 1$$

subject to the conditions

$$a^2+b^2+c^2+f^2+g^2+h^2=1$$

$$a'^2+b'^2+c'^2+f'^2+g'^2+h'^2=1.$$

We have then $\Sigma a^2.\Sigma a'^2-(\Sigma aa')^2=0$

which reduces to $\Sigma(ab'-a'b)^2=0.$ (97.3

Hence, for real values of the quantities, we must have each square separately equal to zero, and the conditions of parallelism are obtained as

$$a=a',\quad b=b',\quad c=c',\quad f=f',\quad g=g',\quad h=h'.$$

98. Theorem: *If two planes are given in general position, there is no direction in either plane which has a parallel in the other.*

Let the guiding lines of the two planes be taken as in § 82.

Any direction in the first plane may be taken as λ_i,

where $\lambda_i=Al_i+Bm_i$ $(i=1, 2, 3, 4)$.

Similarly, any direction in the second plane being taken as μ_i $(i=1, 2, 3, 4)$, we have

$$\mu_i=Cp_i+Dq_i.$$

If the directions λ_i and μ_i are to be parallel, we must have

$$Al_i+Bm_i=k(Cp_i+Dq_i)$$

where k is a multiplier, such that

$$\Sigma(Al_i+Bm_i)^2=k^2(Cp_i+Dq_i)^2$$

or, $$A^2+B^2+2AB(lm)=k^2\{C^2+D^2+2CD(pq)\}$$

whence $$k=\left\{\frac{A^2+B^2+2AB(lm)}{C^2+D^2+2CD(pq)}\right\}^{\frac{1}{2}}.$$

The necessary condition for the co-existence of these relations is obtained by eliminating the parameters A, B, C, D, k between them, in the form

$$\begin{vmatrix} l_1 & m_1 & p_1 & q_1 \\ l_2 & m_2 & p_2 & q_2 \\ l_3 & m_3 & p_3 & q_3 \\ l_4 & m_4 & p_4 & q_4 \end{vmatrix}=0$$

showing that (36.2) the two planes must then lie in the same hyperplane. Hence, there is no direction in either of two planes in general position which is parallel to a direction in the other. This, however, is possible only when the planes intersect in a line, and consequently lie in the same hyperplane. In this latter case, every direction in one parallel to the common line of intersection is parallel to every direction in the other parallel to the same line, and such directions are the only parallel directions.

There are various other peculiar and interesting relations between two or more planes in the fourfold which have no analogues in lower spaces. For further detailed information, the classical works of Veronese * and Segre † should be consulted.

* G. Veronese, *Fondamenti di Geometria*, etc.

† C. Segre, *Mehrdimensionale Raūme*, Encyclop. d. Math Wissen., Bd. III C. 7.

99. Theorem : *Two planes absolutely orthogonal to a third lie in one and the same hyperplane.*

Let $\alpha(\lambda_i, \mu_i)$ and $\beta(\nu_i, \rho_i)$ be two planes absolutely orthogonal to a given plane $\gamma(l_i, m_i)$ at the points $O(a, b, c, d)$ and $O'(a', b', c', d')$ respectively. Let l_i be the line OO'.

The hyperplane determined by α and the point O' is

$$\begin{vmatrix} x-a' & y-b' & z-c' & w-d' \\ l_1 & l_2 & l_3 & l_4 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 \end{vmatrix}=0 \qquad (99.1$$

or, $$L(x-a')+M(y-b')+N(z-c')+P(w-d')=0 \quad \dots \quad (1)$$

where L, M, N, P are the co-factors of $x-a'$, etc., in the determinant, and consequently, are quantities proportional to the direction-cosines of the normal.

If, then, OA and O′B be the normals to this hyperplane at O and O′ respectively, the direction-cosines of OA and O′B are proportional to L, M, N, P.

$\therefore$ OA and O′B lie in one plane, and that plane must be the given plane γ. For, this plane is orthogonal to the hyperplane (1), and consequently to the plane α lying therein, and γ is the only plane orthogonal to α at O, which does not lie in the hyperplane.

$\therefore$ O′B is perpendicular to the lines ν and ρ, so that

$$\Sigma\nu L=0, \qquad \Sigma\rho L=0,$$

showing that the lines ν and ρ both lie in the hyperplane (1); and consequently the plane $\beta(\nu, \rho)$ lies in the same hyperplane (1).

Alternative Method :

We may proceed in a different manner as follows :

From the conditions of orthogonality of the lines we have

$$\Sigma l\lambda=0 \quad \Sigma l\mu=0 \quad \Sigma \nu l=0 \quad \Sigma \rho l=0$$
$$\Sigma m\lambda=0 \quad \Sigma m\mu=0 \quad \Sigma \nu m=0 \quad \Sigma \rho m=0.$$

Eliminating l or m between these equations, we obtain

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 \\ \nu_1 & \nu_2 & \nu_3 & \nu_4 \\ \rho_1 & \rho_2 & \rho_3 & \rho_4 \end{vmatrix}=0 \qquad (99.2$$

showing that, if the four directions are drawn through a point, they lie in one and the same hyperplane (36.2), but the two planes α and β have always a common point, and consequently they lie in one and the same hyperplane.

If the planes α and β are to lie in a hyperplane, a direction L_i can be found which will lie in both the planes. If, therefore, L_i is a direction in the plane α, we must have $\qquad L_i=p\lambda_i+q\mu_i. \qquad \ldots \quad (2)$

Similarly, if L_i is a direction in the plane β, we must have $\qquad L_i=p'\nu_i+q'\rho_i. \qquad \ldots \quad (3)$

These direction-cosines must be proportional and hence

$$\frac{p\lambda_1+q\mu_1}{p'\nu_1+q'\rho_1}=\frac{p\lambda_2+q\mu_2}{p'\nu_2+q'\rho_2}=\frac{p\lambda_3+q\mu_3}{p'\nu_3+q'\rho_3}=\frac{p\lambda_4+q\mu_4}{p'\nu_4+q'\rho_4}=-r \text{ (say)}$$

or, $\qquad p\lambda i+q\mu_i+p'r\nu_i+q'r\rho_i=0 \quad (i=1, 2, 3, 4).$

Eliminating p, q, $p'r$ and $q'r$ between these equations, we obtain the determinant (99.2), which shows that the two planes α and β have a common direction, and consequently they lie in one and the same hyperplane. It is evident that every common point of α and β lies at infinity.

The common direction thus lies at infinity, and α, β are parallel.

Every direction in one plane, parallel to the common direction, is parallel to every direction in the other, parallel to the same common direction, and these directions may be the only parallel directions in the two planes.

Cor.: It follows then that all the planes absolutely perpendicular to any plane at different points of any of its lines lie in a hyperplane and are completely parallel.

Note : It is to be noticed that all the three planes cannot be viewed at the same time. The most that could be seen simultaneously is the plane γ and a line in each of the two planes α and β, or the two planes α and β and the line OO'.

100. Properties of two intersecting Planes:

The following geometrical facts regarding two intersecting planes should be observed:

(1) A plane perpendicular to one of two absolutely perpendicular planes and passing through their common point is perpendicular to the other.

(2) A plane intersecting two absolutely perpendicular planes in lines is perpendicular to both.

(3) If a plane is perpendicular to one of two absolutely perpendicular planes, and contains a point of the other, it is perpendicular to both.

(4) The absolutely orthogonal planes at any common point of two given simply perpendicular planes are themselves simply perpendicular.

(5) Two planes intersecting in a line have one, and only one, pair of common perpendicular planes at any common point.

(6) If two planes have a common perpendicular plane, the absolutely orthogonal plane to the latter is also a common perpendicular plane to the former two.

(7) If two planes intersect in a line, they cannot be absolutely orthogonal.

(8) If two planes meet in a line, the planes through any point respectively absolutely perpendicular to them also meet in a line.

(9) Through a point external to two given planes, there can be drawn only one plane which intersects the given planes in lines.

These and various other similar properties are easily deducible from the analysis explained in §§ 82-92.

101. Theorem: *If three planes have a line in common, their absolutely orthogonal planes at any point of this line lie in a hyperplane; conversely, if three planes with one common point lie in a hyperplane, their absolutely orthogonal planes at this point have a line in common.*

Let the three planes be defined by

$$\left.\begin{array}{lll} \Sigma lx=0, & \Sigma mx=0, & \Sigma nx=0, \\ \Sigma l'x=0, & \Sigma m'x=0, & \Sigma n'x=0 \end{array}\right\} \quad \ldots \quad (1)$$

If these three planes have a common line of section a direction $(\lambda, \mu, \nu, \rho)$ can be determined which will lie in all these six hyperplanes.

$$\therefore \left.\begin{array}{lll} \Sigma l\lambda=0, & \Sigma m\lambda=0, & \Sigma n\lambda=0, \\ \Sigma l'\lambda=0, & \Sigma m'\lambda=0, & \Sigma n'\lambda=0 \end{array}\right\} \quad \ldots \quad (2)$$

Eliminating λ, μ, ν, ρ between these six relations we get

$$\left\| \begin{array}{cccccc} l_1 & l'_1 & m_1 & m'_1 & n_1 & n'_1 \\ l_2 & l'_2 & m_2 & m'_2 & n_2 & n'_2 \\ l_3 & l'_3 & m_3 & m'_3 & n_3 & n'_3 \\ l_4 & l'_4 & m_4 & m'_4 & n_4 & n'_4 \end{array} \right\| = 0. \quad \ldots \quad (3)$$

These are equivalent to 3 different relations between the six directions, and they imply that each group of four of the six lines l, l', m, m', n, n', lie in one and the same hyperplane, and consequently, all the six lie in one and the same hyperplane. But the pairs of lines (l, l'), (m, m'), (n, n') respectively determine the absolutely perpendicular planes of the given planes. Hence, these planes also lie in one and the same hyperplane.

Conversely, if the three planes (1) lie in a hyperplane, their absolutely perpendicular planes will have a common line of section.

For, let the guiding hyperplanes of the three planes be respectively defined by

$$\Sigma lx + k\Sigma l'x = 0, \; \Sigma mx + k'\Sigma m'x = 0, \; \Sigma nx + k''\Sigma n'x = 0. \quad \ldots \quad (4)$$

If the three planes lie in one and the same hyperplane, it will be possible to determine the parameters k, k', k'' so as to identify the three hyperplanes, and

in that case, the co-efficients in (4) will be proportional, *i.e.*, we must have

$$\frac{l_i+kl'_i}{m_i+k'm'_i}=-\lambda \text{ (say)}, \quad \frac{l_i+kl'_i}{n_i+k''n'_i}=-\mu \text{ (say)},$$

$$\frac{m_i+k'm'_i}{n_i+k''n'_i}=-\nu \text{ (say)},$$

$$(i=1, 2, 3\ 4).$$

Eliminating the unknown parameters $k, k', k'', \lambda, \mu, \nu$ between these relations, we obtain

$$\begin{vmatrix} l_1 & l'_1 & m_1 & m'_1 \\ l_2 & l'_2 & m_2 & m'_2 \\ l_3 & l'_3 & m_3 & m'_3 \\ l_4 & l'_4 & m_4 & m'_4 \end{vmatrix}=0, \quad \begin{vmatrix} l_1 & l'_1 & n_1 & n'_1 \\ l_2 & l'_2 & n_2 & n'_2 \\ l_3 & l'_3 & n_3 & n'_3 \\ l_4 & l'_4 & n_4 & n'_4 \end{vmatrix}=0,$$

$$\begin{vmatrix} m_1 & m'_1 & n_1 & n'_1 \\ m_2 & m'_2 & n_2 & n'_2 \\ m_3 & m'_3 & n_3 & n'_3 \\ m_4 & m'_4 & n_4 & n'_4 \end{vmatrix}=0. \qquad \ldots \quad (5)$$

These are, in fact, equivalent to the matrix equation (3), and show that quantities L, M, N, P can be determined so as to satisfy the relations

$$\begin{aligned} &Ll_1+Ml_2+Nl_3+Pl_4=0\\ &Ll'_1+Ml'_2+Nl'_3+Pl'_4=0\\ &Lm_1+Mm_2+Nm_3+Pm_4=0\\ &Lm'_1+Mm'_2+Nm'_3+Pm'_4=0\\ &Ln_1+Mn_2+Nn_3+Pn_4=0\\ &Ln'_1+Mn'_2+Nn'_3+Pn'_4=0 \end{aligned}$$

These relations indicate that the six hyperplanes have a common line of section, and consequently, the three planes determined by them have also a common line of section.

Cor.: If conditions (5) are satisfied, the planes lie in the same hyperplane. We may easily deduce the following:

If three planes intersect in one common line, and lie in one and the same hyperplane, their absolutely orthogonal planes at any common point intersect in a line and lie in one and the same hyperplane.

102. Isoclinal Angle:*

There are always two critical angles ϕ and ψ between two planes in general position (§ 77). The planes of these two angles are common perpendicular to the two given planes and are themselves mutually absolutely orthogonal. We shall now show that any angle θ, formed by any line in one plane with its projection on the other, always lies between the critical angles ϕ and ψ; *i.e.*, if $\phi > \psi$, then $\phi > \theta > \psi$.

Without loss of generality, we may suppose the two planes α and β drawn through the origin as their common point. Let $(l\ ,\ m)$ be the critical lines in α and $(p\ ,\ q)$ the critical lines in β, so that the angle $\widehat{lp}=\phi$ and $\widehat{mq}=\psi$, $\widehat{lm}=\widehat{pq}=$ a rt. angle.

* I. Stringham, *On the Geometry of Planes in a Parabolic Space of Four Dimensions*—Transactions of the Amer. Math. Soc., Vol. 2 (1901), pp. 183-214. Two planes with a constant isoclinal angle are said to be mutually *isoclinal* and Stringham calls two such planes "*isoclines*."

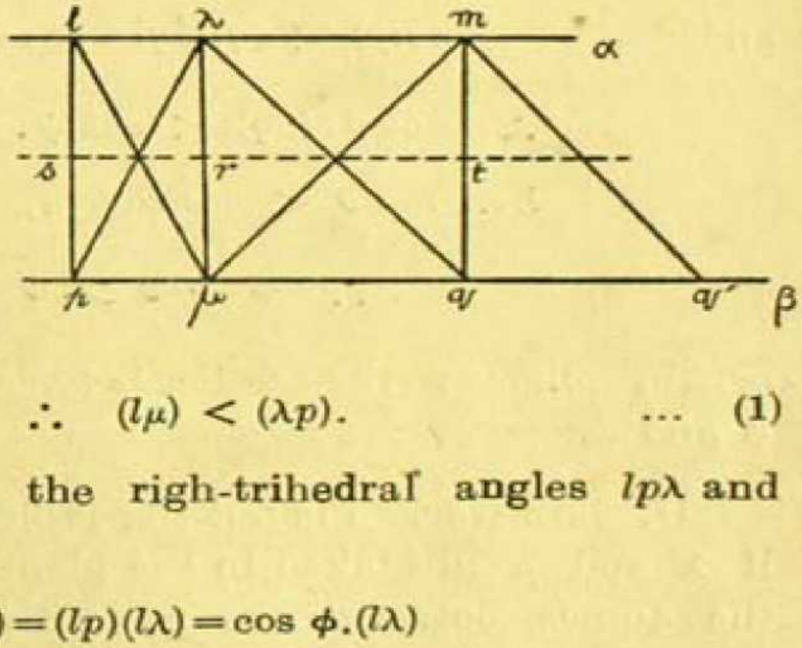
In the plane $\alpha(l, m)$ take a line λ, and let μ be its projection on the plane $\beta(p, q)$, so that $\widehat{\lambda\mu}=\theta$.

In this analysis we will suppose all our lines and planes drawn through the common origin, so that angles between lines may be represented by the arcs of great circles drawn on a hypersphere through the extremities of those lines.

Let $l_i, m_i, p_i, q_i, \lambda_i, \mu_i$ $(i=1, 2, 3, 4)$ denote the direction-cosines of the corresponding lines.

Considering the right trihedral angles $lp\mu$ and $\lambda\mu p$, we have*

$(l\mu)=(lp)(p\mu)$
$=\cos\ \phi.(p\mu)$
and
$(\lambda p)=(\lambda\mu)(p\mu)$
$=\cos\ \theta.(p\mu)$
If, then, $\phi > \theta$,
$\cos\ \phi < \cos\ \theta$.



$$\therefore \quad (l\mu) < (\lambda p). \qquad \dots \quad (1)$$

Again, considering the righ-trihedral angles $lp\lambda$ and $lp\mu$, we have

$$(\lambda p)=(lp)(l\lambda)=\cos\ \phi.(l\lambda)$$

and $$(l\mu)=(lp)(p\mu)=\cos\ \phi.(p\mu)$$

$\therefore$ From (1), $$\cos\ \phi.(l\lambda) > \cos\ \phi.(p\mu)$$

$$\therefore \quad (l\lambda) > (p\mu), \quad i.e., \quad \widehat{l\lambda} < \widehat{p\mu}. \qquad \dots \quad (2)$$

* I. Todhunter, *Spherical Trigonometry*, § 84, p. 55.

Since $\hat{lm} = \hat{pq} =$ a rt. angle, $\lambda\hat{m} > \hat{\mu q}$

or, $$(\lambda m) < (\mu q). \quad \ldots \quad (3)$$

Again, considering the right-trihedral angles λmq and μmq, we have

$$(\lambda q) = (\lambda m)(mq) = (\lambda m) \cos \psi$$

and $$(m\mu) = (\mu q)\ (mq) = (\mu q) \cos \psi.$$

$\therefore$ From (3), we obtain $(\lambda q) < (m\mu)$. ... (4)

Finally, from the right-trihedral angles $\lambda\mu q$ and $m\mu q$, we get

$$(\lambda q) = (\lambda\mu)(\mu q) = \cos\theta \,.\, (\mu q)$$

and $$(m\mu) = (mq)(\mu q) = \cos\psi \,.\, (\mu q)$$

$$\therefore \quad \cos\theta \,.\, (\mu q) < \cos\psi \,.\, (\mu q)$$

$$i.e., \cos\theta < \cos\psi, \text{ or, } \theta > \psi.$$

$$\therefore \quad \phi > \theta > \psi \quad (102.1$$

or, in other words, θ lies between the critical angles ϕ and ψ.

If, however, μ is not the projection of λ on β, but if λ and μ are taken in the planes α and β, dividing the angles between the respective critical lines l, m and p, q in the same ratio, then the plane (λ, μ) makes equal dihedral angles with the planes α and β.

Let χ and χ' be the dihedral angles which the plane (λ, μ) makes with α and β respectively, and let

$$\hat{l\lambda} = \hat{p\mu} \text{ and } \hat{\lambda m} = \hat{\mu q}.$$

If $\lambda_i = Al_i + Bm_i, \quad \mu_i = Ap_i + Bq_i$

$(i = 1, 2, 3, 4)$

Then, $(l\mu) = (lp)(p\mu) = \cos\phi.(p\mu) = \cos\phi.(l\lambda) = (\lambda p)$. ... (5)

Again, $(l\mu) = (l\lambda)(\lambda\mu) + [l\lambda][\lambda\mu].\cos\chi$. * ... (6)

Also, $(\lambda p) = (p\mu)(\lambda\mu) + [p\mu][\lambda\mu].\cos\chi'$

$= (l\lambda)(\lambda\mu) + [l\lambda][\lambda\mu].\cos\chi'$.

$\therefore$ From (5) and (6), we obtain $\cos\chi = \cos\chi'$,

i.e., the plane of (λ, μ) makes equal dihedral angles with α and β.

The angle $\widehat{\lambda\mu}$ has been called the "isoclinal angle" of the planes α and β, the angles ϕ and ψ being its maximum and minimum values.†

103. Theorem: *The plane of the bisectors of the critical angles intersect the plane of the isoclinal angle at right angles in the bisector of the isoclinal angle.*

Let s_i and t_i be the bisectors of the critical angles ϕ and ψ (see Fig., *p.* 181).

Then, $s_i = \frac{1}{2}(l_i + p_i)$ and $t_i = \frac{1}{2}(m_i + q_i)$.

Now, $As_i + Bt_i = \frac{1}{2}\{A(l_i + p_i) + B(m_i + q_i)\}$

$= \frac{1}{2}\{(Al_i + Bm_i) + (Ap_i + Bq_i)\}$

$= \frac{1}{2}(\lambda_i + \mu_i)$. ... (1)

* I. Todhunter—*loc. cit*, § 97, p. 68.

† Stringham, *loc. cit.*

This shows that the line dividing the angle between the lines s_i and t_i in the ratio A : B is a line in the plane (λ, μ), bisecting the isoclinal angle $\hat{\lambda\mu}$. Hence the plane (s_i, t_i) intersects the plane (λ, μ) in the bisector of the isoclinal angle.

Again, if r_i is this bisector, we ha ve

$$(s\lambda) = (sl)(l\lambda) \quad \text{and} \quad (s\mu) = (sp)(p\mu).$$

But $$(sl) = (sp) \quad \text{and} \quad (l\lambda) = (p\mu)$$

$$\therefore \quad (s\lambda) = (s\mu). \qquad \dots \quad (2)$$

Also, $$(s\lambda) = (sr)(\lambda r) + [sr][\lambda r] \cos \chi_1,$$

where χ_1 is the dihedral angle $\angle \lambda rs$,

and $$(s\mu) = (sr).\ (\mu r) + [sr][\mu r] \cos \chi_2$$

where χ_2 is the dihedral angle $\angle \mu rs$.

$$\therefore \quad \cos \chi_1 = \cos \chi_2, \qquad (103.1$$

i.e., the dihedral angles χ_1 and χ_2 are equal, and these being adjacent angles, each of them is a right dihedral angle, and the planes (λ, μ) and (s, t) are mutually perpendicular.

104. Geometry of two Planes with an infinite number of perpendicular Planes:

From what has been stated in § 102, it is now clear that the plane (λ, μ) makes equal dihedral angles with the

given planes α and β and that the isoclinal angle $\widehat{\lambda\mu} \equiv \theta$ lies intermediate between the critical angles ϕ and ψ.

If now $\phi = \psi$, the isoclinal angle θ is equal to each of the equal critical angles, and in this case, the plane (λ, μ) of the isoclinal angle is also a common perpendicular plane of α and β.

From geometry of the ordinary space, we have (see Fig. § 102)

$$(l\mu) = (lp)\ (p\mu) \qquad \dots \quad (1)$$

and

$$(\lambda p) = (lp)\ (l\lambda). \qquad \dots \quad (2)$$

Since $(l\lambda) = (p\mu)$, we have $(l\mu) = (\lambda p)$

i.e., $\widehat{l\mu} = \widehat{\lambda p}$. Similarly, $\widehat{m\mu} = \widehat{\lambda q}$. $\dots$ (3)

We shall denote the dihedral angle between the planes (λ, μ) and (μ, q) by the symbol $\angle\ \lambda\mu q$.

If, now, in β, an angle $\widehat{qq'}$ equal to the angle $\widehat{p\mu}$ is laid off, such that the half-line q' is on the same side of q as μ is of p, then

$$\angle\,\mu q' = \angle\,pq.$$

Again, $(mq') = (mq)\ (qq') = (lp)\ (p\mu)$

$\therefore\ (mq') = (l\mu)$ and $(p\mu) = (qq')$

i.e., $\angle\,mq' = \angle\,l\mu = \angle\,\lambda p$.

Again, $(\lambda q) = (\lambda p)(pq) + [\lambda p][pq] \cos \angle \lambda pq,$

and $(m\mu) = (mq')(\mu q') + [mq'][\mu q'] \cos \angle mq'\mu.$

But $(pq) = (\mu q')$; $(\lambda p) = (mq')$; $(\lambda q) = (m\mu)$

whence,
$$\cos \angle \lambda pq = \cos \angle mq'\mu$$

$\therefore$
$$\angle \lambda pq = \angle mq'\mu,$$

i.e., the dihedral angles along p and q' are equal.

Again,
$$\begin{aligned}(\lambda\mu) &= (\lambda p)(p\mu) + [\lambda p][p\mu] \cos \angle \lambda p\mu \\ &= (mq')(qq') + [mq'][qq'] \cos \angle mq'q \\ &= (mq)\end{aligned}$$

$\therefore$
$$\begin{aligned}(\lambda p) &= (\lambda\mu)(p\mu) + [\lambda\mu][p\mu] \cos \angle \lambda\mu p \\ &= (mq)(qq') + [mq][qq'] \cos \angle \lambda\mu p \\ &= (mq') = (mq)(qq')\end{aligned}$$

$\therefore$ $\cos \angle \lambda\mu p = 0$, *i.e.*, $\angle \lambda\mu p$ is a right dihedral angle.

Thus, the plane (λ, μ) is perpendicular to β, and similarly also perpendicular to α. Hence, the plane (λ, μ) is a common perpendicular plane of both α and β, and the angle $\lambda\mu$ is equal to the equal critical angles ϕ or ψ.

Thus, equal acute angles being laid off within the right angles $\angle lm$ and $\angle pq$, the half-lines thus drawn will determine a plane, which is common perpendicular to the two planes α and β. Consequently, there is an infinite number of such planes.

Conversely, if there be taken two half-lines λ and μ, such that the plane (λ, μ) is perpendicular to both α

and β, then $\angle l\lambda = \angle p\mu$ and $\angle \lambda\mu = \phi = \psi$. For, proceeding exactly as before, it can be proved that

$$\angle \lambda\mu = \angle lp = \angle mq, \text{ which show that } \phi = \psi.$$

Also, since $\angle \lambda p = \angle l\mu$, we have $\angle l\lambda = \angle p\mu$.

Thus, when the critical angles of two planes are equal, there is an infinitude of common perpendicular planes, on each of which they cut out equal angles and the two planes are said to be *isocline*.

Hence, *the condition necessary and sufficient for the two planes being isocline is that their critical angles are equal.*

The above facts may be summarised in the form of the following theorem :—*If two planes cut out equal angles on their two common perpendicular planes, they have an infinitude of common perpendicular planes, on which they cut out equal angles. Any two of these common perpendicular planes again intercept equal angles on the two planes.*

105. Theorem. *If two lines in one plane α make equal angles with another plane β, the line bisecting the angle between them and the line bisecting the angle between their projections upon β will determine a common perpendicular plane of α and β.**

Let l_1 and m_1 be the two lines in α, including an angle ω, and p_1 and q_1 their projections upon β, including an angle ω', so that $\angle lp = \angle mq$.

Let λ_1 be the bisector of the angle lm, and μ_1 that of the angle pq. Then the plane determined by

* This property has been used by Veronese in finding common perpendicular planes—Veronese, *loc. cit*, § 150.

λ_i, μ_i will be a common perpendicular plane of α and β.

Since the dihedral angles along p_i and q_i are right dihedral angles, we have

$$(lq)=(lp)\cos\omega', \quad \text{and} \quad (mp)=(mq)\cos\omega'$$

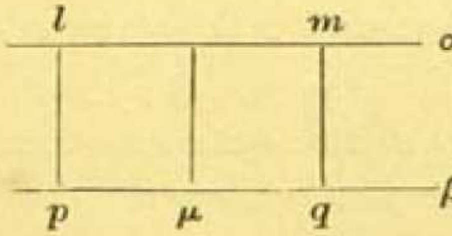
$$\therefore \qquad (lq)=(mp). \qquad \ldots \quad (1)$$

Again, $(l\mu)=(lp)(p\mu)$

and $(m\mu)=(mq)(\mu q)$.

Also $(p\mu)=(\mu q)$, $(lp)=(mq)$

$\therefore$ $(l\mu)=(m\mu)$, *i.e.*, $\angle l\mu=\angle m\mu$.



Again, $(l\mu)=(\lambda\mu)(l\lambda)+[\ \mu][l\lambda]\cos\angle l\lambda\mu$

and $(m\mu)=(\lambda\mu)(m\lambda)+[\lambda\mu][m\lambda]\cos\angle m\lambda\mu$.

$\therefore$ The dihedral angle $\angle l\lambda\mu$ = dihedral angle $\angle m\lambda\mu$. These, being adjacent angles, are each a right dihedral angle, *i.e.*, the plane (λ, μ) is perpendicular to α.

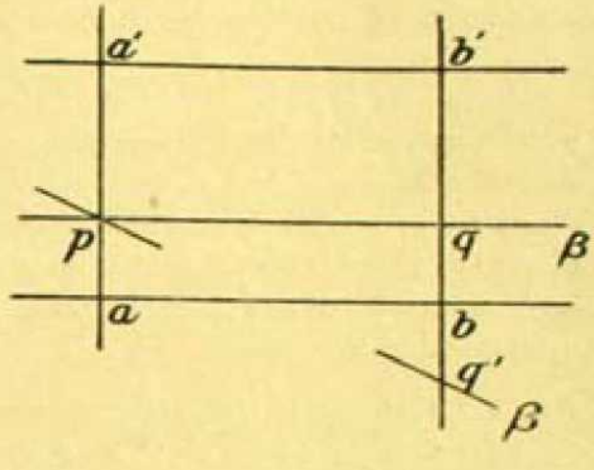
Proceeding as above, it can be shewn that the plane (λ, μ) is also perpendicular to the plane β. When α and β are isocline, the angle $\angle\lambda\mu$ is equal to the two equal critical angles, and consequently, there is an infinitude of common perpendicular planes, and conversely.

106. Construction of a Plane making given critical Angles with another.

The orthogonal frame of axes is often very useful in constructing planes with given critical angles.

Let α be a given plane. It is required to construct a plane β making critical angles ϕ and ψ with α.

Let a and b be two perpendicular lines in the plane α, and let a' and b' be two perpendicular lines in the absolutely perpendicular plane α', so that the four lines a, b, a', b' form four mutually perpendicular axes of an orthogonal frame.



Now, in the plane aa', draw the line p making $\angle ap = \phi$ and in the plane bb', draw the line q making $\angle bq = \psi$.

The plane determined by p, q is then the required plane. For, the planes aa' and bb' are absolutely perpendicular, and the plane $\beta(p, q)$ meets them both in lines. Hence, the plane $\beta(p, q)$ is perpendicular to both aa' and bb' (§ 100), *i.e.*, the planes aa' and bb' are common perpendicular planes of both α and $\beta(p, q)$, and consequently, the angles $ap \equiv \phi$ and $bq \equiv \psi$ are the critical angles between α and $\beta(p, q)$.

If q' be drawn in the plane bb' on the other side of b, so that the angle bq' is now $-\psi$, the plane pq' is obtained making critical angles ϕ and $-\psi$ with the given plane α.

When $\phi = \psi$, the plane $\beta(p, q)$ is isocline to α, as well as to α'. By giving different values to ϕ, a series of planes, such as (p, q), can be constructed, which are all isocline to α and α' and to one another. Similarly, when $\phi = -\psi$, we can construct another series of planes

all isocline to α and α', but in a different sense (to be explained in the next chapter).

Note : It is to be noted that there is a sense of rotation which turns p through $90°$ to the position of q. The angles between α and pq as constructed are ϕ and ψ, but when the angle ψ is increased by π, the plane (q, p) is obtained, which makes angles ϕ and $\psi+\pi$ with α. The same plane (p, q) will, however, be obtained, if each of ϕ and ψ be increased by any multiple of 2π, or if both be increased simultaneously by odd multiples of π.

Ex. 1. Calculate the critical angles between the planes $x_1+x_2=0$, $x_3-x_4=0$; $x_1-x_2=0$, $x_3+x_4=0$.

Ex. 2. Shew that the plane $x_1+2x_2+1=0$, $x_3-x_4+2=0$ is both half-parallel and half-orthogonal to the plane $2x_1+x_2=0$, $x_1-2x_3=0$.

Ex. 3. Shew that the two planes $x_1=0$, $x_2+x_3+x_4=0$ and $x_3+x_4=0$, $x_1+x_2=0$ intersect in a line.

Ex. 4. Shew that the plane (a, b, c, a, b, c') is isocline to the plane $x_1=x_2=0$, the isoclinal angle being $\cot^{-1}\sqrt{-c/c'}$.

Ex. 5. Show that the plane $z=1+x+2y$, $w=3x-y$ is parallel to the plane $z=x+2y$, $w=2+3x-y$.

Ex. 6. Prove that the planes drawn absolutely orthogonal to a given plane at different points along a line are parallel to one another.

Ex. 7. The critical angles between two planes are $\frac{\pi}{6}$ and $\frac{\pi}{4}$; show that a circle in one of these planes projects on to the other in an ellipse of eccentricity $\frac{1}{\sqrt{3}}$.

CHAPTER VII

MOTION IN THE FOURFOLD

107. Displacement in Geometry:

We are all familiar with the movements of material bodies in the ordinary space, but these movements are incapable of representation in geometry, and in fact the idea of motion is foreign to geometry, although we often speak of a point, or a line moving in a plane or a space. What is really meant by this is that the attention for the time being is fixed on a particular point or line, and that is successively shifted from one position to the next, giving the impression of a continuous motion. When we say that a figure is displaced in space, we really mean that our attention is transferred from one figure to another congruent to the former, or what is the same thing, we consider a transformation in which certain relations—distances, angles, etc., remain invariant. Thus, the idea of motion in geometry is principally based on the notions of *order* and *congruence*. In plane geometry there are two corresponding groups of axioms, which may be extended to higher geometries by means of proper definitions. There is no scope for their detailed discussion in the present work, for which the reader may consult the works of Forder, Hilbert and Manning.*

* H. G. Forder, *The Foundations of Euclidean Geometry* (1927). D. Hilbert, *The Foundations of Geometry*—translated by E. J. Townsend (1902). H. P. Manning, *Geometry of Four Dimensions* (1927), §§ 88-96, pp. 153-167.

108. General Motion :

The general motion of a material body in the ordinary space is of two varieties, namely, (1) translation, by which the body moves parallel to itself in a fixed direction, (2) rotation, by which it can move around a fixed point or a line. Correspondingly, in geometry two varieties of motion are recognised :

(1) a figure is moved without deformation from one position to another, *i.e.*, parallel to itself, each of its points moving in parallel directions;

(2) a figure is rotated without deformation about a fixed point or a fixed line.

In a plane there is only one type of rotation, namely, rotation around a point, which is called the *centre* of rotation. In a hyperplane, there are two types of rotation, namely, (1) rotation around a point, (2) rotation about a line, which is called the *axis of rotation*. Thus, a plane perpendicular to a line at any point O can rotate on itself around O, remaining always perpendicular to the line, or it may rotate around any line parallel to it or about one of its own lines as the axis-line. Since there is much greater freedom of motion in the fourfold than in our space, there is a new type of rotation, namely, rotation about a plane, the analogue of which is not to be found in the ordinary space. We shall be presently engaged in studying this new type of rotation in the fourfold. A hyperplane (or a material body) can rotate around any plane or about any of its own planes. This fixed plane is called the *axis-plane* of rotation and the rotation is called a *simple rotation*.

In geometry, however, we are concerned with the motion of geometrical figures, and in particular, with points in the fourfold. The general motion in the fourfold is uniquely determined by the motion of any four non-conjoint points, *i.e.*, a motion which carries any four non-conjoint points of a figure from one position to another will carry the entire figure from the first to the second position; for, each point of the hyperplane determined by the four points comes to this second position by the motion. Any point not belonging to the hyperplane remains at the same distance from it, on the same side of it and with the same projection upon it. Hence, the entire figure comes from the original to the second position by the same motion.

From the analytical stand-point, it may be argued that the scheme of general orthogonal transformation in four variables involves twenty independent parameters. The condition that any four non-conjoint points transform to four other non-conjoint points is equivalent to sixteen relations among these parameters. The fact that the four assumed points are non-conjoint supplies four additional conditions, namely, the conditions for the constancy of the mutual distances of the four points. Thus, the twenty parameters in the scheme of transformation are connected by twenty independent equations of condition, and are, therefore, uniquely determinable. The transformation formulae thus obtained will transform any point to the corresponding point in the new position.

109. Diverse Types of Rotation :

From what has been stated above, it is seen that in the fourfold, there are three distinct types of rotations :—

(1) A rotation can leave a point unaltered, while around this point there is a general displacement without deformation. This may be called *rotation around a point.*

(2) A rotation can leave a line unchanged, while around it, a general displacement without deformation can take place. It may be called an *axial rotation.* The displacement under rotation round a line is more restricted than that under rotation around a point, but it is of the same nature as the general displacement around a point in space, and the movement of every point is in a plane at right angles to the axis of rotation.

(3) A rotation can leave a plane unaltered, while around it there is a general displacement without deformation in every orthogonal plane. It may be called a rotation around a plane or a *simple rotation.* The general displacement under this rotation is still more restricted and is of the same nature as the orthogonal displacement of the axes in a plane, orthogonal to the plane round which the rotation takes place.

There is, however, no rotation which leaves a hyperplane unaltered.

The geometric theory of rotation in the fourfold has been given in a memoir by Van Oss * and the

* Van Oss, *The Elementary Motion of Space S_4*—Mém de l'Ac d'Amsterdam Seance du, 21 November, 1901.

analytic theory in a paper by Cole * and the dissertation by Ganguli. †

110. Rectangular Frame of Axes :

The analytical theory of rotation in the fourfold can conveniently be studied with the help of a rectangular system of axes. As already explained, four mutually perpendicular lines drawn through the origin O form a rectangular frame. Distinguishing positive and negative directions along these lines, let a, b, c, d be four positive half-lines through O forming a rectangular system. Any set of three of these lines will form a rectangular system of axes in a hyperplane normal to the fourth. Without disturbing the fourth this orthogonal frame can be changed from one position to another by a suitable rotation round a duly chosen axis, for instance, they can be permuted cyclically by a rotation around a half-line equally inclined to these three. Thus, the four axes can be permuted in twelve different ways, giving twelve congruent arrangements. On the other hand, any arrangement of three of these half-lines may be obtained by a rotation in the fourfold.

111. Rotation represented by orthogonal Transformation :

Let l_i, m_i, n_i, p_i $(i=1, 2, 3, 4)$ be the direction-cosines of four mutually perpendicular lines through

* F. N. Cole, *On Rotations in Space of Four Dimensions*, American Journal of Mathematics, Vol. XII (1890).

† S. M. Ganguli, *Analytical Geometry of Hyperspaces*, Vol. 2, Chap. II (1914).

the origin, forming a new rectangular frame. Let $P(x', y', z', w')$ be a point referred to this frame, whose co-ordinates referred to the original frame are (x, y, z, w). The formulae of transformation are then given by

$$\left.\begin{aligned}
x' &= l_1 x + m_1 y + n_1 z + p_1 w \\
y' &= l_2 x + m_2 y + n_2 z + p_2 w \\
z' &= l_3 x + m_3 y + n_3 z + p_3 w \\
w' &= l_4 x + m_4 y + n_4 z + p_4 w
\end{aligned}\right\} \tag{111.1}$$

The direction-cosines of the new axes are connected by ten equations of condition, namely,

$$\begin{gathered}
\Sigma l_1{}^2 = 1, \quad \Sigma l_2{}^2 = 1, \quad \Sigma l_3{}^2 = 1, \quad \Sigma l_4{}^2 = 1, \\
\Sigma l_1 l_2 = 0, \quad \Sigma l_2 l_3 = 0, \quad \Sigma l_3 l_4 = 0, \quad \Sigma l_1 l_4 = 0, \\
\Sigma l_2 l_4 = 0, \quad \Sigma l_3 l_1 = 0.
\end{gathered} \tag{111.2}$$

Since the origin remains unaltered, the equation of the hypersphere $x^2 + y^2 + z^2 + w^2 = R^2$ will be transformed into $x'^2 + y'^2 + z'^2 + w'^2 = R^2$, in virtue of the relations (111.2).

Instead of considering a change of axes, we may view this from a different and more general standpoint, namely, that of rotation of the hypersphere about its centre into itself, and in this view of the matter, rotation in the fourfold may be analytically represented by a general orthogonal transformation, analogous to what is met in the lower geometries. In the general orthogonal transformation, the sixteen constants used are connected by the ten relations (111.2). Consequently, these constants are capable of being expressed in terms of six independent others and the

corresponding rotation about a point consists of ∞^6 different operations.

If a, b, c, f, g, h are six independent constants, using Cayley's formula,* the sixteen co-efficients can be expressed as follow :

$$kl_1 = 1 + f^2 + g^2 + h^2 - a^2 - b^2 - c^2 - \triangle^2$$

$$km_1 = 2(a + \triangle f - bh + cg)$$

$$kn_1 = 2(b + \triangle g - cf + ah)$$

$$kp_1 = 2(c + \triangle h - ag + bf)$$

$$kl_2 = 2(-a - \triangle f + cg - bh)$$

$$km_2 = 1 + f^2 + b^2 + c^2 - a^2 - g^2 - h^2 - \triangle^2$$

$$kn_2 = 2(h + \triangle c + fg - ab)$$

$$kp_2 = 2(-g - \triangle b + hf - ac)$$

$$kl_3 = 2(-b - \triangle g - cf + ah)$$

$$km_3 = 2(-h - \triangle c + fg - ab)$$

$$kn_3 = 1 + g^2 + c^2 + a^2 - b^2 - h^2 - f^2 - \triangle^2$$

$$kp_3 = 2(f + \triangle a + gh - bc)$$

$$kl_4 = 2(-c - h\triangle + bf - ag)$$

$$km_4 = 2(g + \triangle b + fh - ac)$$

$$kn_4 = 2(-f - \triangle a + gh - bc)$$

$$kp_4 = 1 + h^2 + a^2 + b^2 - c^2 - f^2 - g^2 - \triangle^2$$

where $\triangle \equiv af + bg + ch,$

and $k \equiv 1 + a^2 + b^2 + c^2 + f^2 + g^2 + h^2 + \triangle^2.$

* Cayley, Crelle's Journal, Vol. XXXII.

112. Axis-plane of a simple Rotation :

When the (centre) origin remains invariant, the above transformation transforms the hypersphere into itself, or what is the same thing, it represents the rotation of the fourfold around a fixed point (the origin). Let us now examine if any other points besides the origin can remain invariant under this transformation. In that case, the co-ordinates of the invariant points must satisfy the relations (111.1) when x', y', z', w' are replaced by x, y, z, w respectively, *i.e.*, they must satisfy the relations

$$\left.\begin{aligned} (l_1-1)x+m_1y+n_1z+p_1w=0 \\ l_2x+(m_2-1)y+n_2z+p_2w=0 \\ l_3x+m_3y+(n_3-1)z+p_3w=0 \\ l_4x+m_4y+n_4z+(p_4-1)w=0 \end{aligned}\right\} \qquad (112.1$$

for values of x, y, z, w other than 0, 0, 0, 0.

The necessary condition for this is obtained by eliminating the variables between the same equations in the determinant form

$$\begin{vmatrix} l_1-1 & m_1 & n_1 & p_1 \\ l_2 & m_2-1 & n_2 & p_2 \\ l_3 & m_3 & n_3-1 & p_3 \\ l_4 & m_4 & n_4 & p_4-1 \end{vmatrix}=0 \qquad (112.2$$

Replacing the l's, m's, n's, p's by their values calculated above, the determinant reduces to $\triangle^2/k$.

Hence, we conclude that if other points besides the origin are to remain invariant, we must have

$$\Delta^2 = 0. \quad i.e., \quad \Delta \equiv af + bg + ch = 0 \tag{112.3}$$

which is a condition necessary for other points remaining invariant.

When this condition is satisfied, the four relations (112.1) reduce to two independent equations, namely,

$$\left.\begin{aligned} (a^2+b^2+c^2)x-(a-bh+cg)y-(b-cf+ah)z \\ -(c-ag+bf)w=0 \\ (a-cg+bh)x+(a^2+g^2+h^2)y-(h+fg-ab)z \\ -(-g+fh-ac)w=0 \end{aligned}\right\} \tag{112.4}$$

Thus, it appears that the invariant points lie on the locus defined by the equations (112.4), which is evidently a plane, *i.e.*, the points of the plane defined by (112.4) remain invariant under the scheme of transformation.

We conclude therefore that, subject to the condition (112.3), the general orthogonal transformation is equivalent to a rotation of the fourfold, in which a plane remains unaltered, or what is the same thing, the rotation takes place around a fixed plane as an axis-plane, which, as stated before, is a possible operation in the fourfold, having no analogue in lower spaces.

A rotation which leaves a plane unaltered is called a *simple rotation* and the fixed plane is called the *axis-plane* of the rotation.

It can easily be shewn that the orientation-cosines of the plane (112.4) are respectively

$$a^2k^2,\ abk^2,\ ack^2,\ afk^2,\ agk^2 \text{ and } ahk^2,$$

where $$k = 1 + a^2 + b^2 + c^2 + f^2 + g^2 + h^2.$$

Hence, the orientation-cosines of the axis-plane are proportional to a, b, c, f, g, h, which are the six independent constants used in the scheme of transformation.

113. The Angle of a simple Rotation:

From what has been stated above, it is easily seen that the axis-plane is not only converted into itself but its individual points also remain unaltered. All other points in the fourfold rotate around this plane, and the rotation takes place in planes absolutely perpendicular to the axis-plane. Each of these planes of rotation is converted into itself without deformation, *i.e.*, the plane rotates as a whole on itself around the point where it meets the axis-plane, the mutual relative positions of the individual points remaining unchanged. In fact, each absolutely perpendicular plane rotates about the point through a certain angle, which may be called the *angle of rotation*.

The rotation of a solid body about any of its planes is a strange phenomenon, not to be experienced in the ordinary space. It is therefore necessary to further elucidate the operation by means of other explanatory statements.

Suppose the plane α separates a hyperplane into two parts S and T. At any point O of α draw a perpendicular to α, such that the part OA lies in S and

the part OB lies in T. At each point of α we can similarly draw a unique perpendicular. The total number of such perpendiculars are then equal to the number of points in α, *i.e.*, ∞^2. These perpendiculars rotate about the points in the following manner:

If the two half-hyperplanes S and T are rotated in the same sense, leaving the plane α fixed, the line AOB rotates about the point O, always remaining perpendicular to α, and thus generates a plane β absolutely perpendicular to α. When AOB rotates through an angle $180°$, half of the plane β is described and the half-hyperplanes S and T interchange positions. If now AOB is again rotated through an angle of $180°$, the other half of β is described and the half-hyperplanes S and T again occupy their original positions, that is to say, the hyperplane has described the entire fourfold. In this motion then, the plane α remains absolutely fixed, the hyperplane, and in fact, the fourfold, turns as a wheel around the point O, and all such absolutely perpendicular planes as β are rotated through the same angle, which is called the *angle of rotation*.

114. The Angle of Rotation in terms of the Parameters of the Axis-Plane:

Any plane is uniquely determined by four conditions, but the orientation-cosines of the plane of rotation are connected by the single relation

$$\Delta \equiv af + bg + ch = 0,$$

and are consequently equivalent to five independent parameters. Thus, one degree of freedom is left to the

plane of rotation, and the angle of rotation is, therefore, a function of a single parameter.

If we take the orientation-cosines of the plane as

$$a \cot\frac{\theta}{2}, \quad b \cot\frac{\theta}{2}, \quad c \cot\frac{\theta}{2}, \quad f \cot\frac{\theta}{2}, \quad g \cot\frac{\theta}{2}, \quad h \cot\frac{\theta}{2},$$

we have

$$(a^2+b^2+c^2+f^2+g^2+h^2)=\tan^2\frac{\theta}{2} \qquad (114.1$$

or $$1+a^2+b^2+c^2+f^2+g^2+h^2=1+\tan^2\frac{\theta}{2}=\sec^2\frac{\theta}{2}=k.$$

We may arrive at the same result from a different consideration.

Since the plane of rotation rotates on itself about a point, any line in this plane drawn through the point will rotate through the angle of rotation θ. We may select a normal to the axis-plane, for instance, the normal to the first hyperplane (112.4) which evidently lies in the plane of rotation.

The direction-cosines are given by

$$\lambda : \mu : \nu : \rho=(a^2+b^2+c^2) : (-a+bh-cg)$$
$$:(-b+cf-ah):(-c+ag-bf)$$

whence, $$\lambda=-(a^2+b^2+c^2)/\vartheta,$$

$$\mu=(a-bh+cg)/\vartheta,$$

$$\nu=(b-cf+ah)/\vartheta$$

$$\rho=(c-ag+bf)/\vartheta,$$

where $\vartheta\equiv(a^2+b^2+c^2)(1+f^2+g^2+h^2+a^2+b^2+c^2)$,

If $(\lambda', \mu', \nu', \rho')$ be the directions-cosines of the new assigned position of the line after rotation,

$$\cos\theta = \lambda\lambda' + \mu\mu' + \nu\nu' + \rho\rho'.$$

Substituting λ, μ, ν, ρ respectively for x, y, z, w in the scheme (111.1), the values of $\lambda', \mu', \nu', \rho'$, which are proportional to x', y', z', w', can be obtained.

The values of $\lambda', \mu', \nu', \rho'$ thus calculated will give the value of θ, after simplification, in the form

$$\cos\theta = \frac{(a^2+b^2+c^2)k^2 - 2(a^2+b^2+c^2)(k^2-k)}{(a^2+b^2+c^2)k^2}$$

$$= \frac{2k-k^2}{k^2}$$

$$= \frac{2-k}{k}, \text{ where } k \equiv 1+f^2+g^2+h^2+a^2+b^2+c^2$$

$$= \frac{1-f^2-g^2-h^2-a^2-b^2-c^2}{1+f^2+g^2+h^2+a^2+b^2+c^2}. \tag{114.2}$$

We easily find that

$$\frac{1-\cos\theta}{1+\cos\theta} = f^2+g^2+h^2+a^2+b^2+c^2$$

i.e., $$\tan^2\frac{\theta}{2} = f^2+g^2+h^2+a^2+b^2+c^2 \tag{114.3}$$

which agrees with the previous result.

115. Displacement of a rectangular Frame :

From the foregoing analysis of simple rotation in the fourfold, it is easily seen that a rectangular frame of four axes can be changed from any position to any other, without change of origin, by two

suitable rotations around two absolutely perpendicular planes, the two rotations being independent of each other.

A rotation round an axis-plane leads to a rotation of the axes of reference in the absolutely perpendicular plane. Hence, simultaneous independent rotations around two absolutely perpendicular planes lead to simultaneous independent rotations of the axes of reference in the two planes. Thus, when we take the six co-ordinate planes in three orthogonal pairs, three distinct operations are possible, each consisting of a rotation round one plane through any angle and of a simultaneous rotation round the absolutely perpendicular plane through any other angle different from the first.

These operations may be effected in the following order:

(1) A rotation round the plane YOZ through an angle α' and a simultaneous rotation round the plane XOW through an angle α, where $\alpha \neq \alpha'$.

If a point (x, y, z, w) has now the new co-ordinates (x_1, y_1, z_1, w_1) we have

$$\left.\begin{aligned} x_1 &= x \cos \alpha' + w \sin \alpha' \\ y_1 &= y \cos \alpha + z \sin \alpha \\ z_1 &= -y \sin \alpha + z \cos \alpha \\ w_1 &= -x \sin \alpha' + w \cos \alpha' \end{aligned}\right\} \qquad (115.1$$

(2) A rotation round the plane ZOX through an angle β' and a simultaneous rotation round the plane YOW through an angle β, where $\beta \neq \beta'$. A point (x, y, z, w) will now have co-ordinates (x_2, y_2, z_2, w_2),

where

$$\left.\begin{aligned} x_2 &= x \cos\beta - z \sin\beta \\ y_2 &= y \cos\beta' + w \sin\beta' \\ z_2 &= x \sin\beta + z \cos\beta \\ w_2 &= -y \sin\beta' + w \cos\beta' \end{aligned}\right\} \qquad (115.2$$

(3) A rotation round the plane XOY through an angle γ' and a simultaneous rotation round the plane ZOW through an angle γ, where $\gamma \neq \gamma'$. A point (x, y, z, w) will now have co-ordinates (x_3, y_3, z_3, w_3)

where

$$\left.\begin{aligned} x_3 &= x \cos\gamma + y \sin\gamma \\ y_3 &= -x \sin\gamma + y \cos\gamma \\ z_3 &= z \cos\gamma' + w \sin\gamma' \\ w_3 &= -z \sin\gamma' + w \cos\gamma' \end{aligned}\right\} \qquad (115.3$$

If then, owing to the displacement of the orthogonal frame defined by the scheme (111.1), a point$(x, y, z, w,)$ occupies now a new position (x', y', z', w'), it is always possible to choose three pairs of rotations successively around three selected pairs of absolutely orthogonal planes, so that the final displacement may be obtained by the combination of these three displacements; and these may be so arranged that the six constants α, α', β, β', γ, γ' are all independent.* Consequently, the sixteen co-efficients used in the above scheme are functions of these six independent parameters, as has been otherwise found before. These six parameters, which are

* For details see Forsyth, *Geometry of Four Dimensions*, Vol. I, pp. 188-192 (1930).

functions of the angles of rotation of the three pairs of component rotations, are found to fix the axis-plane of the resultant and are really proportional to the orientation-cosines of the axis-plane of the simple rotation defined by (111.1).

116. Axial Rotation in the Fourfold:

It is now easy to see that in the fourfold rotation around a point, taken to be the origin, is composite and can be compounded of rotations round planes. In fact, in all rotations in the fourfold, the fundamental elements can be made to rotate around a plane, in which there is a displacement of perpendicular axes in the absolutely perpendicular plane.

In an axial rotation, *i.e.*, in a rotation around a line, there is the most completely free movement of the hyperplane normal to the axis, since every direction in the hyperplane is unrestricted in every way except that it is perpendicular to the axis. Hence, all displacements in a hyperplane, keeping the origin fixed and preserving the orthogonality of the orthogonal frame, may be represented as an axial rotation. Now then, in such a rotation, the axis as well as the normal to the hyperplane are fixed, and consequently, the plane determined by these two lines remain fixed as a whole. The axial rotation, therefore, when considered as a rotation in the fourfold, is really a rotation around a plane. Hence, every axial rotation in the fourfold can be regarded as a rotation around an axis-plane. This can very easily be inferred from what has been said in § 112, the condition $\Delta = 0$ for an axial rotation also implying the rotation around a plane.

117. Jordan's Theorem on small Displacement: *

What has been stated above about displacement of an orthogonal frame also holds, even when the displacement is very small, *i.e.*, the transformation of the frame can be effected by two small rotations round two absolutely perpendicular planes, and these rotations are mutually independent.

If the origin remains invariant, a small displacement means only a small change in the defining co-ordinates and a corresponding change in the equations of transformation, which now require to be so modified as to exhibit the small change.

Let the scheme of transformation be given as in (111.1). Since the change is small, the quantities $x'-x$, $y'-y$, $z'-z$, $w'-w$ are also small. Again, since the co-efficients are all constants, each of l_1, m_2, n_3, p_4 must be nearly equal to unity, *i.e.*, (say) $1-\epsilon_1$, $1-\epsilon_2$, $1-\epsilon_3$, $1-\epsilon_4$, where ϵ_1, ϵ_2, ϵ_3, ϵ_4, are small quantities. The remaining twelve co-efficients must also be small quantities.

Now, $$l_1{}^2+m_1{}^2+n_1{}^2+p_1{}^2=1,$$

or, $$(1-\epsilon_1)^2+m_1{}^2+n_1{}^2+p_1{}^2=1$$

i.e., $$-2\epsilon_1+\epsilon_1{}^2+m_1{}^2+n_1{}^2+p_1{}^2=0,$$

shewing that ϵ_1 is a small quantity of order higher than m_1, n_1, p_1. Rejecting, then, small quantities of orders higher than the first, ϵ_1 may be taken to be 0; and by a similar reasoning, we can assume $\epsilon_2=0$, $\epsilon_3=0$,

* C. Jordan, *Essai sur la géométrie à n dimensions—Bull. de la Soc. de France*, Vol. VIII (1875), §98, pp. 152-171.

$\epsilon_4=0$; *i.e.*, for small displacements, retaining small quantities only of the first order, we may take

$$l_1=1, \quad m_2=1, \quad n_3=1, \quad p_4=1.$$

Again, from the relation

$$l_1l_2+m_1m_2+n_1n_2+p_1p_2=0,$$

since n_1n_2 and p_1p_2 are small quantities of order higher than the first, rejecting these we get

$$l_1l_2+m_1m_2=0, \text{ or, } l_2+m_1=0.$$

Similarly, from the remaining five conditions of orthogonality, we obtain, when the transformation is infinitesimal,

$$l_3+n_1=0, \quad l_4+p_1=0, \quad m_3+n_2=0,$$
$$m_4+p_2=0, \quad n_4+p_3=0.$$

If now we put

$$m_1=-l_2=a, \quad n_1=-l_3=h, \quad p_1=-l_4=g,$$
$$n_2=-m_3=b, \quad p_2=-m_4=f, \quad p_3=-n_4=c,$$

so that a, b, c, f, g, h are six independent small quantities of the first order of smallness, they are sufficient to represent a small displacement, which may now be represented by the scheme:

$$\left.\begin{aligned} x'&=x+ay+hz+gw \\ y'&=-ax+y+bz+fw \\ z'&=-hx-by+z+cw \\ w'&=-gx-fy-cz+w \end{aligned}\right\} \qquad (117.1$$

where a, b, c, f, g, h are independent small quantities of the first order of smallness, satisfying the conditions of orthogonality.

118. Combination of simple Rotations :

Theorem : *Two simple rotations around two different axis-planes can be combined into a simple rotation around an axis-plane, if, and only if, the two axis-planes lie in one and the same hyperplane.*

Let the component rotations be defined by

$$\left.\begin{aligned} x'_i &= a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + a_{i4}x_4 \\ x''_i &= a'_{i1}x'_1 + a'_{i2}x'_2 + a'_{i3}x'_3 + a'_{i4}x'_4 \end{aligned}\right\} \tag{118.1}$$

$(i=1, 2, 3, 4)$.

If the two rotations take place in the order in which they are written, we have

$$\begin{aligned} x''_i &= a'_{i1}(a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4) \\ &\quad + a'_{i2}(a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4) \\ &\quad + a'_{i3}(a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4) \\ &\quad + a'_{i4}(a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4) \\ &\equiv a''_{i1}x_1 + a''_{i2}x_2 + a''_{i3}x_3 + a''_{i4}x_4 \end{aligned}$$

where $a''_{ij} = a'_{i1}a_{1j} + a'_{i2}a_{2j} + a'_{i3}a_{3j} + a'_{i4}a_{4j}$

$(i=j=1, 2, 3, 4)$.

Let a, b, c, f, g, h and $a', b', c' f', g', h'$ be the orientation-cosines of the two component axis-planes, and let

$$\Delta \equiv af + bg + ch, \qquad \Delta' \equiv a'f' + b'g' + c'h',$$

and
$$\Delta'' \equiv a''f'' + b''g'' + c''h''.$$

If $a'', b'', c'', f'', g'', h''$ denote the six independent constants, in terms of which the co-efficients of the

resultant transformation can be expressed, then we have*—

$$\left.\begin{aligned}
\mathrm{D}a''&=a+a'-\Delta'f-\Delta f'+bh'-b'h+c'g-cg'\\
\mathrm{D}b''&=b+b'-\Delta'g-\Delta g'+cf'-c'f+a'h-ah'\\
\mathrm{D}c''&=c+c'-\Delta'h-\Delta h'+ag'-a'g+b'f-bf\\
\mathrm{D}f''&=f+f'-\Delta'a-\Delta a'+bc'-b'c+gh'-g'h\\
\mathrm{D}g''&=g+g'-\Delta'b-\Delta b'+ca'-c'a+f'h-fh'\\
\mathrm{D}h''&=h+h'-\Delta'c-\Delta c'+ab'-a'b+fg'-f'g
\end{aligned}\right\}\qquad(118.2$$

where $\mathrm{D}\equiv 1+\Delta\Delta'-aa'-bb'-cc'-ff'-gg'-hh'$

$$\therefore\quad \mathrm{D}\Delta''=\Delta+\Delta'+a'f+af'+bg'+b'g+ch'+c'h. \qquad (118.3$$

Since the components are simple rotations, we must have

$$\Delta=\Delta'=0$$

$\therefore$ It follows from (118.3) that

$$\mathrm{D}\Delta''=a'f+af'+b'g+bg'+c'h+ch'.$$

If, then, the resultant is to be a simple rotation, Δ'' must be zero, and this requires that

$$a'f+af'+b'g+bg'+c'h+ch'=0. \qquad (118.4$$

This last condition implies that the axis-planes of the two component rotations intersect in a line (92.2), which proves the proposition.

Hence, if the resultant is to be a simple rotation, the necessary condition is that *the axis-planes of the two component rotations should intersect in a line, i.e., they should lie in one and the same hyperplane.*

* These may be calculated from Cayley's formula cited before. See also Cole's paper referred to.

That this is a sufficient condition also follows from the fact that when

$$\triangle = \triangle' = 0$$

and also

$$af' + a'f + b'g + bg' + c'h + ch' = 0, \quad \text{then} \quad \triangle'' = 0,$$

and the resultant is consequently a simple rotation.

Note : The resultant of two simple rotations is not, in general, a simple rotation. The condition, both necessary and sufficient, that the resultant will be a simple rotation is that the axis-planes of the two component rotations should lie in the same hyperplane.

Cor. : The resultant of two simple rotations around two mutually absolutely orthogonal planes cannot be a simple rotation. It is called a *double rotation*.

119. Theorem : *The axis-plane of the resultant contains the line of intersection of the axis-planes of the two component rotations, or the three axis-planes are all perpendicular to one and the same plane.*

Let $a_k,\ b_k,\ c_k,\ f_k,\ g_k,\ h_k\ (k=1,\ 2,\ 3)$

denote the orientation-cosines of the three axis-planes. It is then easy to shew, by actual calculation, that*

$$a_1f_2 + a_2f_1 + b_1g_2 + b_2g_1 + c_1h_2 + c_2h_1 = 0,$$
$$a_2f_3 + a_3f_2 + b_2g_3 + b_3g_2 + c_2h_3 + c_3h_2 = 0,$$
and $$a_1f_3 + a_3f_1 + b_1g_3 + b_3g_1 + c_1h_3 + c_3h_1 = 0,$$

showing that the three planes intersect two by two in lines.

* See Author's *Geometry of Hyperspaces*, Vol. II, § 23, p. 39.

Suppose the axis-planes are respectively defined by the pairs of normals (l_i, l'_i); (m_i, m'_i); (n_i, n'_i) and they intersect in the common line (λ_i).

Then, from the conditions of orthogonality, we obtain

$$\left.\begin{array}{lll} \Sigma\lambda_i l_i=0, & \Sigma\lambda_i m_i=0, & \Sigma\lambda_i n_i=0, \\ \Sigma\lambda_i l'_i=0. & \Sigma\lambda_i m'_i=0, & \Sigma\lambda_i n'_i=0. \end{array}\right\} \quad \ldots \quad (1)$$

From the first two, we get (§ 33)

$$\lambda_1 L_{1j}+\lambda_2 L_{2j}+\lambda_3 L_{3j}+\lambda_4 L_{4j}=0 \quad (j=1,\ 2,\ 3,\ 4) \quad \ldots \quad (2)$$

Similarly, from the second and the third pairs, we get

$$\lambda_1 M_{1j}+\lambda_2 M_{2j}+\lambda_3 M_{3j}+\lambda_4 M_{4j}=0 \quad \ldots \quad (3)$$

$$\lambda_1 N_{1j}+\lambda_2 N_{2j}+\lambda_3 N_{3j}+\lambda_4 N_{4j}=0 \quad \ldots \quad (4)$$

where L's are a_1, b_1, c_1, etc.; M's are a_2, b_2, c_2, etc.; N's are a_3, b_3, c_3, etc.

Eliminating λ_1, λ_2, λ_3, λ_4 between these equations, we obtain

$$\left\Vert\begin{array}{cccc} L_{1j} & L_{2j} & L_{3j} & L_{4j} \\ M_{1j} & M_{2j} & M_{3j} & M_{4j} \\ N_{1j} & N_{2j} & N_{3j} & N_{4j} \end{array}\right\Vert=0,\ (j=1,\ 2,\ 3,\ 4).$$

These give four independent conditions ; and when a_1, b_1, c_1,...etc., are substituted for L's, M's, N's respectively, the result may be expressed in one matrix equation

$$\left\Vert\begin{array}{cccccc} a_1 & b_1 & c_1 & f_1 & g_1 & h_1 \\ a_2 & b_2 & c_2 & f_2 & g_2 & h_2 \\ a_3 & b_3 & c_3 & f_3 & g_3 & h_3 \end{array}\right\Vert=0. \qquad (119.1$$

Hence, if these conditions are satisfied by the orientation-cosines of the three axis planes, they have a common line of intersection.

On the contrary, when these conditions are not satisfied, six independent constants a, b, c, f, g, h may be determined satisfying the seven conditions:

$$af+bg+ch=0$$

$$\left.\begin{array}{l} a_k f+b_k g+c_k h+af_k+bg_k+ch_k=0 \\ a_k a+b_k b+c_k c+f_k f+g_k g+h_k h=0 \end{array}\right\} \quad (k=1, 2, 3)$$

showing that there is a plane with orientation-cosines a, b, c, f, g, h, which intersects each of the three axis-planes orthogonally, and this proves the proposition.

120. Rotations around two absolutely orthogonal Planes are commutative:

Theorem: *Rotations round two absolutely orthogonal planes are commutative; after the two rotations, points of the fourfold occupy the same position, in whichever order the rotations are performed.*

Let a, b, c, f, g, h be the orientation-cosines of any plane; then those of its absolutely perpendicular plane are f, g, h, a, b, c respectively.

If $a_1, b_1, c_1, f_1, g_1, h_1$ be the parameters of transformation in the resultant, then by using the formulae (118.2), we have, since $a' \equiv f$, $b' \equiv g$, $c' \equiv h$, $f' \equiv a$, $g' \equiv b$, $h' \equiv c$ and $\triangle = \triangle' = 0$,

$$Da_1 = a+f+bc-gh+gh-bc = a+f.$$

Similarly,

$$Db_1=b+g, \qquad Dc_1=c+h, \qquad Df_1=f+a,$$

$$Dg_1=g+b, \qquad Dh_1=h+c,$$

$$D=1-2(af+bg+ch)=1,$$

$$\begin{aligned}\Delta''=a_1f_1+b_1g_1+c_1h_1&=(a+f)^2+(b+g)^2+(c+h)^2\\ &=a^2+b^2+c^2+f^2+g^2+h^2+2(af+bg+ch)\\ &=a^2+b^2+c^2+f^2+g^2+h^2.\end{aligned}$$

Since the parameters in the resultant are symmetrical as regards the elements of the two component rotations, it follows that the two rotations can be effected in any order, the resultant being always the same.

It is to be noted that for real values of a, b, c, etc., Δ'' can never vanish, and consequently the resultant can never be a simple rotation. It is a *double rotation* around two absolutely perpendicular planes.

121. Symmetry in the Fourfold :

DEFINITION : Two points are said to be symmetrically situated with regard to the mid-point of the line joining the points, and the mid-point is called the *centre of symmetry*.

Two points are symmetrically situated with regard to a line, a plane or a hyperplane which is perpendicular to the line joining the points at their centre of symmetry.

When the corresponding points of two figures are so situated, the figures are said to be *symmetrical*.

In the fourfold, figures may be symmetrical with regard to a point, a line, a plane or a hyperplane.

Theorem 1 : *Two figures symmetrically situated with regard to a plane can be brought into position of coincidence by a simple rotation of one of them through 180° around this plane.*

The relation between a body and its image on a looking glass affords an apt illustration of two figures symmetrical with regard to a plane. The image clearly appears as if the body is just standing behind the mirror, but the misleading impression will at once be dispelled by observing that when you raise your right hand, the image will be found to raise its left hand. This phenomenon is explained by saying that the body and the image are *symmetrically* situated with regard to the plane of the mirror. The two may be made to coincide, point for point, by rotation of one of them through 180° around the plane of the mirror.

Considering the plane of the mirror as the co-ordinate plane (x, y), the truth of the theorem will follow from the formulae (115.3) by putting $\gamma'=180°$ and $\gamma=0$. The points (x, y, z, w) and $(x, y, -z, -w)$ are evidently symmetrical with regard to the plane (x, y).

Theorem 2: *If a figure be rotated through 180° around each of two absolutely orthogonal planes, it will occupy a new position symmetrical to the first with regard to the common point of the two axis-planes of rotation.*

The truth of this theorem will immediately follow from the formulae (115.1) by taking the planes (y, z) and (x, w) as the axis-planes, and putting $\alpha=\alpha'=180°$, when the point (x, y, z, w) is found to be transformed to the point $(-x, -y, -z, -w)$, symmetrical with regard to the origin.

122. Motion of a Hypersphere :

The centre remaining fixed, any position of a hypersphere can be obtained from any other by a simple or a double rotation.

In a simple rotation around an axis-plane through the centre, the hypersphere is transformed into itself. Therefore, when it is rotated successively around two absolutely orthogonal axis-planes through the centre, it will be transformed into itself, *i.e.*, the hypersphere will remain unaltered as a whole, but the individual points, except those along the axis-planes, will be displaced, without any change in their mutual distances.

This fact suggests that if a portion of its boundary-content is moved freely along the boundary, it will wholly coincide with the boundary itself. Thus, it appears that a hypersphere is a space of constant positive curvature.* If the space we live in were a hypersphere in a euclidean fourfold, we should realise what is called Elliptic Geometry, in which our space is assumed to be a space of constant curvature like a hypersphere and not a space of no curvature like a hyperplane. In fact, the geometry of the hypersphere is the same as the double Elliptic geometry.†

123. The Notion of complex Angle :

Since rotation around a plane is a possible phenomenon in the fourfold, it is now clear that any two planes in general position, which always intersect in a

* Gauss, *Coll. Works*, Bd. IV, pp. 219-58.

† Manning, *Non-Euclidean Geometry*.

point, may be brought into coincidence by means of the two following operations:

If α, β be two planes in general position with the critical angles ϕ and ψ between them:

(1) A simple rotation of one plane β about the plane of ϕ through an angle ψ will make the two planes α and β intersect in a line, *i.e.*, in the line in which the critical lines, the arms of the angle ψ, will coincide; and consequently α and β will now lie in the same hyperplane and the plane of ϕ meets both α and β orthogonally, so that ϕ becomes the dihedral angle between α and β.

(2) The plane β may again be rotated around its line of section with α through an angle ϕ, which will make the two planes coincide, or what is the same thing, a second simple rotation of β around the plane absolutely perpendicular to that of ϕ, *i.e.*, around the plane of ψ, which now passes through the common line, through an angle ϕ will bring the two planes into coincidence.

Thus, it is clear that two planes in general position can be brought into coincidence by two successive simple rotations around two axis-planes (of ϕ and of ψ) which are mutually absolutely orthogonal, *i.e.*, the planes of ϕ and ψ are the axis-planes of these two simple rotations. It is to be noted, however, that the two rotations are commutative, *i.e.*, the rotations can take place in any order, the result being the same.

Thus, for defining the inclination of two planes in general position in the fourfold, we recognise that it must be dependent on these two simple rotations. For an accurate expression, therefore, a term should be used, which could combine the notion of the two simultaneous but successive rotations, and the term *complex*

angle seems appropriate to indicate the double rotation which will bring the two planes into coincidence, *i.e.*, a "complex angle" may be defined as the angle formed by two planes in general position in the fourfold, and we say that the two planes include between them a "complex angle," indicating thereby that the operation required to make the two planes coincide is not a *simple* but a *double rotation*, one followed by another and the word "complex" is sufficiently expressive of this fact. The word further suggests that the double rotation consists of two parts —one a rotation through a plane-angle about a point and the other a rotation through a dihedral angle about a line. It is further interesting to note that the planes of these two component rotations are mutually absolutely orthogonal.

The appropriate notation for denoting a complex angle may consequently be used in the form $\omega=\phi+i\psi$, after Argand's notation for expressing a complex quantity of the form $z=x+iy$. The notation clearly indicates that the complex angle ω consists of two parts, ϕ and ψ, which are essentially real, and the symbol 'i' stands for expressing the fact that the plane of ψ is absolutely orthogonal to that of ϕ. Thus, a generalised geometrical representation of a complex quantity $x+iy$ seems possible, and it may be made to represent any complex quantity—the component parts being linear or angular magnitudes. The quantity $i\psi$ may be taken to represent an imaginary angle, in view of the fact that its plane does not lie in the same hyperplane with that of ϕ.

With this conventional restriction, it is perfectly logical to postulate that the angle between two

planes in general position in the fourfold is a "complex angle," consisting of two parts—one a real part ϕ and the other an imaginary part ψ. It should be noted, however, that any one of them may be taken to represent the real part and the other is then to be regarded as constituting the imaginary part.

124. Analogy to Argand's Diagram:

The above conventional notation is analogous to the representation of a complex quantity in an Argand's plane. Two planes in general position in the fourfold, mutually absolutely orthogonal, can be taken to represent two co-ordinate planes of reference—one of them being called the plane of reals (z) and the other the plane of imaginaries (w). Take any real angle ϕ in the real plane (z) with arms (l) and (m), and in the imaginary plane (w), take an angle ψ with arms (p) and (q). Then the planes (l, p) and (m, q) will include a complex angle $\phi + i\psi$. It is interesting to note that, according to the convention adopted, the two planes (l, q) and (m, p) at the same time are found to include a complex angle $\phi + i\psi$ between them. In fact, it will be noticed that any angle ϕ in the plane of z, combined with any angle ψ in the plane of w, will determine, as above, two pairs of planes, each of which may be regarded as forming an angle $\phi + i\psi$. The two planes forming a complex angle $\phi + i\psi$ between them may be called the "faces" of the angle. It is easily seen that unlike Argand's representation of a complex quantity, the representation of a complex angle for given values of ϕ and ψ is not unique, but admits of doubly infinite pairs of representation.

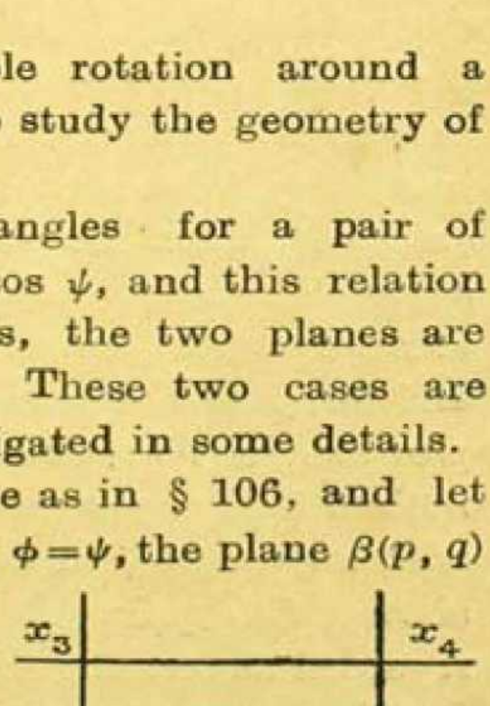
However remarkable may be the formal analogy in this representation, there is the difficulty of locating a real axis-plane for the resultant and ascertaining the magnitude of the rotation, owing to the fact that the resultant of two simple rotations around two absolutely orthogonal planes is not a simple rotation, and as such there is no real plane angle (§120) associated with the two planes whose magnitude may be taken to represent the absolute measure of the complex angle. Thus, this mode of representation may be interesting from the point of view of formal analogy but nothing more.

125. Senses of Isoclinism :

Having discussed the simple rotation around a plane, it is now made possible to study the geometry of isocline planes in more details.

If ϕ and ψ are the critical angles for a pair of isocline planes, we have $\cos\phi = \cos\psi$, and this relation is satisfied when $\phi = \pm\psi$. Thus, the two planes are isocline, when, $\phi = +\psi$, or, $-\psi$. These two cases are distinct and require to be investigated in some details.

Consider the orthogonal frame as in § 106, and let (x_1, x_2) be a given plane α, when $\phi = \psi$, the plane $\beta(p, q)$ is isocline to α, but when $\phi = -\psi$, the plane β' (p, q') is also isocline to α, making the same *isoclinal* angle with it. Thus, the two planes (p, q) and (p, q'), although both isocline to α, are not of the same species, and should,



therefore, be distinguished from each other by a *sense* corresponding to the two possible arrangements of the orthogonal frame. We say that the plane $\beta(p, q)$ is isocline in *one sense* and $\beta'(p, q')$ is isocline in the *opposite sense*. When the former is denoted as *positive*, the latter may be denoted as *negative* and *vice versa*.

Thus, a plane may be isocline to another in two senses—one *positive* and the other *negative*. When two planes are isocline to a given plane in opposite senses, one is said to be *positively* (or directly) isocline and the other is said to be *negatively* (or inversely) isocline.*

In the above construction, if ϕ is measured in the planes (x_1, x_3), and (x_2, x_4), from x_1 and x_2 respectively, both in the same direction towards x_3 and x_4, or away from them, *i.e.*, if it be measured in opposite directions in both, we obtain two planes isocline to α in the same sense. If, however, ϕ is measured in one direction in one of the planes (x_1, x_3), (x_2, x_4) and in opposite direction in the other, we obtain two planes isocline to α in the opposite sense.

126. Reversing the construction of § 125, if we start with the planes (p, q) and (p, q') intersecting in the line p, the plane (x_1, x_2) may be constructed in two ways.

* The reader may consult a paper by R. C. Bose, *New methods in Euclidean geometry of four dimensions*, Bull., Calcutta Math. Soc., Vol., 17 (1926), pp. 105-140, where the author has given some analytical discussion of isocline planes.

Through p draw the plane (x_1, x_3) perpendicular to the planes (p, q) and (p, q'). Then its absolutely perpendicular plane is also perpendicular to both and intersect them in lines q and q' respectively. Let x_2 be the bisector of the angle qq'. If now x_1 be drawn in the plane (x_1, x_3), making $\angle px_1 = \angle qx_2$, the lines x_1 and x_2 determine a plane to which (p, q) and (p, q') are isocline in opposite senses. If, however, $\angle px_1$ is measured in the direction opposite to that in which $\angle qx_2$ is measured, we obtain the plane (x_1', x_2) to which again (p, q) and (p, q') are isocline in opposite senses. Thus, starting with the planes (p, q) and (p, q'), we have constructed two planes (x_1, x_2) and (x_1', x_2) (and also their absolutely perpendicular planes) to which (p, q) and (p, q') are isocline in opposite senses. Hence we obtain the following

Theorem: *Given two intersecting planes, two pairs of absolutely perpendicular planes can be determined, to each of which the given planes are isocline in opposite senses.*

127. Some Properties of isocline Planes:

From what precedes, it is now evident that the isoclinal angle, when $\phi = \psi$, has neither a maximum nor a minimum value, since it always lies between the two extremes. We, therefore, say that any two isocline planes have a (constant) *isoclinal* angle. They have an infinite number of (transversal) common perpendicular planes (§ 84) on which they cut out equal constant angles. Several remarkable properties of

isocline planes are immediately deducible from the foregoing analysis. The following, among others, may be mentioned for ready reference :*

(1) If a plane α is isocline to another plane β, then β is also isocline to α in the same sense. (One may be constructed from the other in exactly the same manner.)

(2) Any of their transversal planes makes with them equal dihedral angles.

(3) Any two planes isocline to a third in one sense are themselves isocline (in the same sense).

(4) Through a given straight line, there can always be passed two planes isocline to a given plane (one positively and the other negatively)—The planes (p, q), (p, q') are isocline to (x_1, x_2).

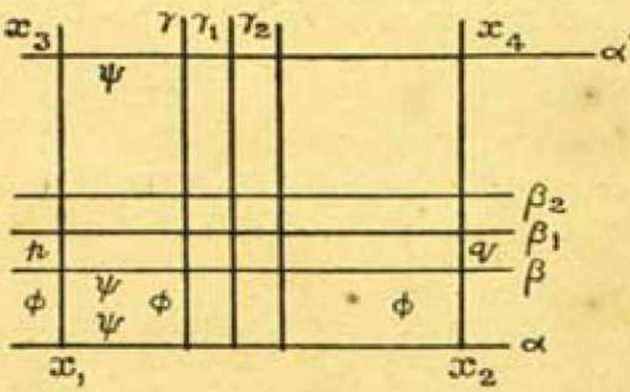
128. Conjugate Series of isocline Planes :

It has already been established that two isocline planes have an infinite number of common perpendicular planes on which they cut out the same angle ϕ, and any two of these common perpendiculars again cut out the same angle ψ on both the given planes.

If, then, α and β are two isocline planes inclined at an angle ϕ, and if an angle ψ is measured from x_1 and p respectively, the terminal half-lines l and m will then specify a plane γ orthogonal to both α and β. The plane γ is evidently orthogonal to α' also, and the

* A geometrical treatment of the properties of isocline planes in the Fourfold has been given by H. P. Manning, *Geometry of four dimensions*, §§ 104-112, pp. 180-198, which may profitably be consulted for further elucidation.

lines l and m include an angle ϕ in γ and determine a sense of rotation corresponding to $x_1\ x_3$. Any line in γ making an angle of $+90°$ with l will certainly lie in the plane $\alpha'(x_3, x_4)$ and make an angle ψ with x_3. It is seen, then, that the



plane γ may be regarded as being determined by laying off equal angles ψ in α and α'. Since this construction has no reference to the angle ϕ or the position of the plane α, the plane γ is orthogonal to all the planes of the isocline series $\beta, \beta_1, \beta_2 \ldots$, obtained by giving different values to ϕ.

Again, giving different values to ψ, an infinitude of isocline planes $\gamma_1, \gamma_2, \cdots$, all orthogonal to all the planes of the isocline series $\beta, \beta_1, \beta_2, \cdots$, can similarly be constructed. The planes of the γ-series are constructed exactly as the isocline β-series is constructed, and themselves form an isocline series, having planes of the β-series for common orthogonal planes. It is to be noted that no two planes of the same series have a common line of intersection, but a plane of one (γ) series intersects all the planes of the other (β) series. In fact, if a plane meets *three* planes of an isocline series in straight lines, it meets them all in straight lines.

Thus, two series of isocline planes, each plane of either series being orthogonal to all planes of the other series, are obtained, which may be called *Conjugate Series of Isocline Planes*.

129. Diverse isocline Series :

The isocline series $\beta, \beta_1, \beta_2, \ldots,$ is not unique. Besides this, as constructed above, there are diverse series of planes isocline to α, not belonging to the β-series. The construction of these series may be effected as follows :

Suppose α is taken as the axis-plane of a simple rotation. Then, α' will be rotated on itself through a certain angle (angle of rotation) θ. The terminal lines x_3 and x_4 will now occupy new positions in α', both orthogonal to x_1 and x_2. A new orthogonal frame will thus be obtained, in which the lines x_1 and x_2, as well as the plane α, remain unaltered. If with this new orthogonal frame, we repeat the previous construction, a new series of planes $\delta_1, \delta_2\ \delta_3, \ldots,$ isocline to α, will be obtained with a new series of common perpendicular planes (conjugate planes) $\epsilon_1, \epsilon_2, \epsilon_3 \ldots$. It is to be noted that no plane of these latter series is orthogonal to any plane of the $\beta, \beta_1, \beta_2, \ldots$ series, except the two planes α and α'.

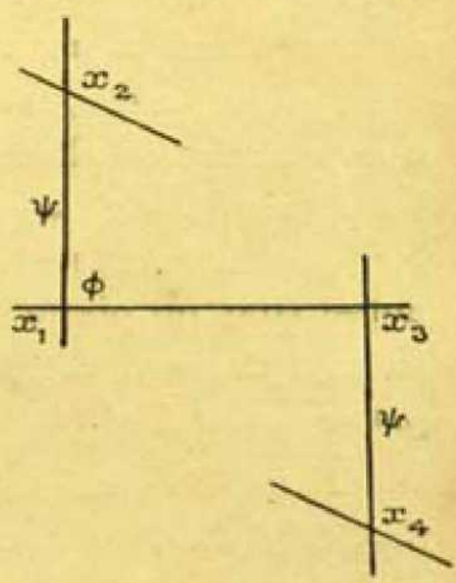
Thus, the series $\delta_1, \delta_2, \delta_3 \ldots,$ distinct from the series $\beta, \beta_1, \beta_2, \ldots,$ is isocline to α. In fact, the former series as a whole may be obtained from the latter by a simple rotation around α, the planes of the two systems converting themselves generally into one another.

130. Theorem : *Two conjugate series of isocline planes are isocline in opposite senses.*

The senses of isoclinism of two series of isocline planes may be well compared by placing one series in

the position of the other with a common plane, to which the latter may be isocline. To compare the senses of isoclinism of the β-series and the γ-series, we apply a simple rotation, which will bring the γ-series into the position of the β-series, the new γ-series now being isocline to α. In fact, if the orthogonal frame is rotated through 180° around the plane specified by the half-line x_1, and the half-line bisecting the angle $\widehat{x_3x_2}$, after rotation, x_3 will come over to the position x_2, and *vice versa*, and the positions of the planes (x_1, x_3) and (x_1, x_2) will be interchanged.

The direction of x_4 will now be reversed, and the plane (x_3, x_4) will occupy the position of (x_4, x_2) and not (x_2, x_4). It is now easily seen that one angle ψ, originally measured in the direction x_1x_2, will have to be measured in the new direction (x_1x_2) (in the direction of ϕ in the original figure), but the other angle ψ, which was measured along x_3x_4, is now to be measured in a direction opposite to that in which the angle ϕ was measured in the original position, *i.e.*, the β-series is obtained by measuring ϕ in one direction in the original figure, while the γ-series is now obtained by measuring ψ in the opposite direction, or in other words, *two conjugate series of isocline planes are isocline in opposite senses.*

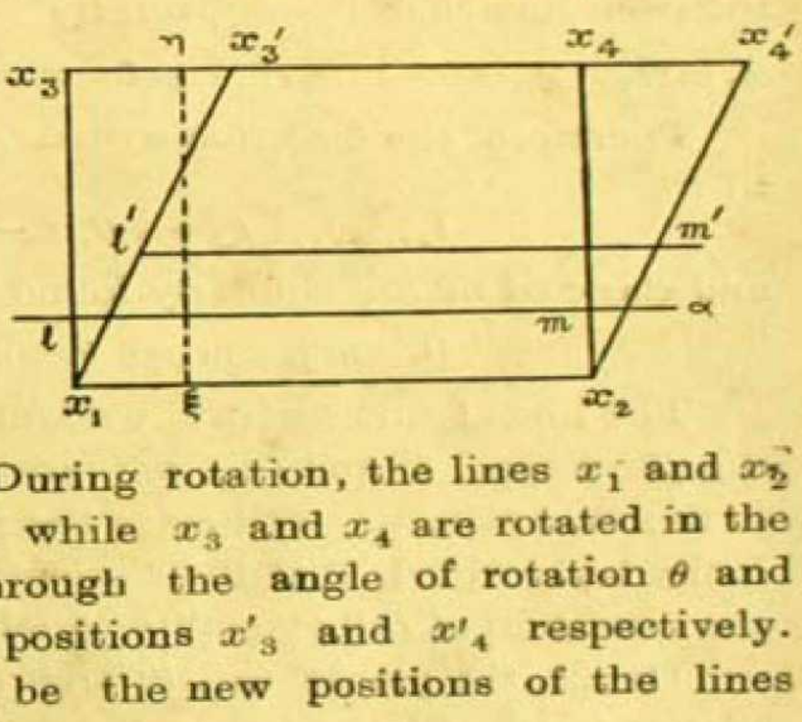


Cor.: Two absolutely perpendicular planes are isocline in both senses.

131. Rotation of conjugate Series of isocline Planes:

A series of planes isocline to a given plane α, which may be taken as the axis-plane of a simple rotation, remains isocline to it and to one another even after rotation and the conjugate series is rotated through the angle of rotation about the lines in which the planes intersect the axis-plane.

Without loss of generality, we may take the co-ordinate plane (x_1, x_2) of an orthogonal frame as the axis-plane of a simple rotation. Let α be a plane isocline to the plane (x_1, x_2) with the critical lines l, m and x_1, x_2 respectively. Then the co-ordinate planes (x_1, x_3) and (x_2, x_4) belong to the conjugate series.



During rotation, the lines x_1 and x_2 remain invariant, while x_3 and x_4 are rotated in the plane (x_3, x_4) through the angle of rotation θ and occupy (say) new positions x'_3 and x'_4 respectively.

Let l' and m' be the new positions of the lines l and m respectively.

It is now required to be shown that the plane (l', m') is still isocline to the plane (x_1, x_2) and that the plane (x_1, x'_3) of the conjugate series is orthogonal to the plane (x_1, x_2) and that the dihedral angle between the planes (x_1, x_3) and (x_1, x'_3) is equal to the angle of rotation θ.

The formulæ of transformation corresponding to the rotation are* :—

$$\left.\begin{aligned} \xi_1 &= x_1, \qquad \xi_2 = x_2, \\ \xi_3 &= x_3 \cos\theta \;\; + x_4 \sin\theta, \\ \xi_4 &= -x_3 \sin\theta + x_4 \cos\theta. \end{aligned}\right\} \qquad \ldots \quad (1)$$

The direction-cosines of the lines l and m can be taken as

$$(l_1, 0, l_3, 0) \qquad \text{and} \qquad (0, m_2, 0, m_4).$$

If then $(rl_1, 0, rl_3, 0)$ be the co-ordinates of any point l on l, which is transformed to l' on the line l', the co-ordinates of l' are given by

$$x_1 = rl_1, \;\; x_2 = 0, \;\; x_3 = rl_3 \cos\theta, \;\; x_4 = -rl_3 \sin\theta. \;\ldots\; (2)$$

Therefore, the direction-cosines of l' are proportional to

$$l_1, \quad 0, \quad l_3 \cos\theta, \quad -l_3 \sin\theta$$

and those of m' are similarly found to be proportional to

$$0, \quad m_2, \quad m_4 \sin\theta, \quad m_4 \cos\theta.$$

The lines l' and m' are evidently orthogonal, and the planes (x_1, l') and (x_2, m') are both perpendicular to the axis-plane as well as to the plane (l', m'), as can easily be verified that their orientation-cosines satisfy the condition of orthogonality (§ 92).

Hence, the planes (x_1, l') and (x_2, m') are critical planes for the planes (x_1, x_2) and (l', m').

Evidently, $\cos \angle \widehat{x_1 l'} = \dfrac{l_1}{\sqrt{l_1^2 + l_3^2 \cos^2\theta + l_3^2 \sin^2\theta}}$

$$= \frac{l_1}{\sqrt{l_1^2 + l_3^2}} = l_1, \text{ since } l_1^2 + l_3^2 = 1.$$

* Salmon, *Conic Sections*, § 9.

Similarly, $\cos \angle x_2 m' = m_2$.

Now, since the plane (l, m) is isocline to (x_1, x_2),

$$l_1 = m_2.$$

$$\therefore \quad \cos \angle x_1 l' = \cos \angle x_2 m', \quad i.e., \quad \angle x_1 l' = \angle x_2 m',$$

or in other words, the plane (l', m') is still isocline to (x_1, x_2) in the same sense, as the plane (l, m) is isocline to (x_1, x_2), *i.e.*, the plane (l, m) remains isocline in the same sense even after rotation.

If χ is the dihedral angle between the planes (x_1, l) and (x_1, l'), we have (from § 82)

$$\cos \chi = \frac{(ll') - (lx_1)(l'x_1)}{[lx_1]\,[l'x_1]} = \frac{(ll') - (lx_1)^2}{[lx_1]^2} = \frac{(ll') - l_1^2}{[lx_1]^2}$$

where $\cos \widehat{ll'} = (ll'), \quad \sin \widehat{ll'} = [ll']$.

But $(ll') = l_1^2 + l_3^2 \cos \theta$

$$\therefore \quad \cos \chi = \frac{(ll') - l_1^2}{1 - l_1^2} = \frac{l_1^2 + l_3^2 \cos \theta - l_1^2}{l_3^2} = \cos \theta. \quad (131.1$$

i.e., the dihedral angle χ = the angle of rotation θ

= constant.

Hence, we conclude that *the planes of the conjugate series are rotated through the angle of rotation about the lines in which they intersect the axis-plane.*

Since the angle χ is constant, it is clear that the two planes of the two conjugate series through the same line x_1 make a constant dihedral angle along the line x_1, and this is true for any line in the axis-plane (x_1, x_2).

This may be stated in the form of the following theorem: *If two planes are isocline in the same sense to a third plane, then two conjugate planes (common perpendicular planes) through any line of this latter form a constant dihedral angle for all positions of the line.*

It may also be shown that if the two planes have with the third a single pair of common perpendicular planes, perpendicular to all three, then the two planes belong to the same system of isocline planes.

It follows further that *the dihedral angles formed by corresponding planes of two conjugate series at their intersections are equal.*

132. Two conjugate Series to a given Series:

From what has been said above, the series (x_1, x_3) and (x_1, x'_3) are both conjugate to the series (x_1, x_2). If they are both to be isocline to any plane, from the construction of § 125 it follows that this plane meets (x_1, x_2) and (x_3, x_4) in ξ and η respectively, such that $\angle x_1\xi = \angle x_3\eta = \angle \eta x'_3$. Consequently, the planes (x_1, x_3) and (x_1, x'_3) are isocline to (ξ, η) in opposite senses. They are also isocline to the absolutely perpendicular plane of (ξ, η).

Again, the planes (x_1, x_2) and (x_3, x_4) are common perpendicular planes of all the three planes (x_1, x_3), (x_1, x'_3) and (ξ, η). From what has been said in § 128, it can be easily recognised that these are the only planes which are perpendicular to all the three.

All these results can be expressed in the form of the following

Theorem : *Two isocline series conjugate to a given series are isocline to another series in opposite senses*, or what is the same thing, *a series of isocline planes can be found to which two series, conjugate to a third, are isocline in opposite senses. These three series have one and only one pair of common perpendicular planes.*

Or, in other words, *if two planes are isocline to a third in opposite senses, the three planes have one and only one pair of common perpendicular planes.*

133. **Theorem** : *Two planes which are isocline to a third in the same sense are also isocline to each other in the same sense.*

It is to be noted that when two planes are said to be isocline to a third, it is understood that they do not generally belong to the same series of isocline planes.

The truth of this theorem easily follows from what has been said in § 132.

That the two planes (l, m) and (l', m'), which are isocline to the plane (x_1, x_2) in the same sense, are isocline can be seen at once. For,

$$(ll') = l_1^2 + l_3^2 \cos\theta \quad \text{and} \quad (mm') = m_2^2 + m_4^2 \cos\theta.$$

Since $\quad l_1 = m_2 \quad$ and $\quad l_3 = m_4, \quad$ we have

$$(ll') = (mm'), \qquad i.e., \qquad \angle ll' = \angle mm'.$$

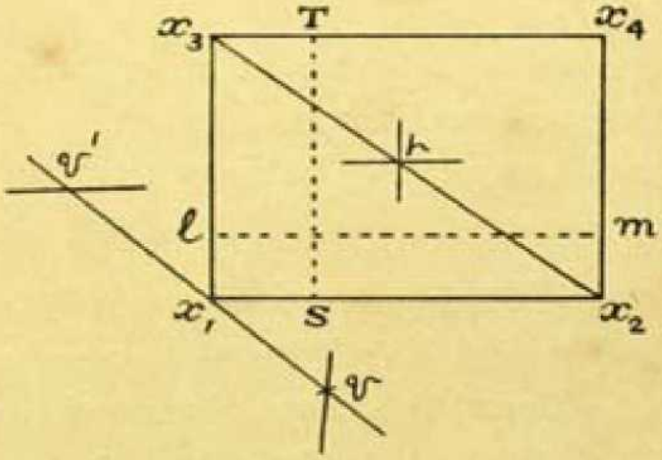
That the planes (l, l') and (m, m') are common perpendicular of the planes (l, m) and (l', m') can be

shown from the fact that the orientation-cosines satisfy the necessary conditions of orthogonality.

The planes (l, m) and (l', m') are isocline to each other in the sense in which each of them is isocline to the plane (x_1, x_2), otherwise they will have one pair of common perpendicular planes, perpendicular to all three, or, they will belong to the same series of isocline planes, which is impossible.

134. Theorem: *There is a pair of absolutely perpendicular planes to which two conjugate series of isocline planes are isocline in opposite senses, at an angle of 45°.*

Consider two conjugate series of planes of the types $\alpha(x_1, x_2)$ and $\beta(x_1, x_3)$. These planes then intersect at right angles, and from what has been stated in § 126, it follows that there is a pair of absolutely perpendicular planes to which the planes of the two series are isocline in opposite senses, at an angle equal to half the dihedral angle between the planes. Since the planes of the two series intersect mutually at right angles, they are isocline in opposite senses at an angle of 45°.



This pair of planes may be constructed as follows:

There are two (mutually absolutely perpendicular) planes which are orthogonal to both the intersecting

planes $\alpha(x_1, x_2)$ and $\beta(x_1, x_3)$, one of these planes (q, q') passes through their common line x_1 and the other is the plane (x_3, x_2), which measures the right dihedral angle between them.

Bisect the angle $x_3 x_2$ by the line p. In (q, q') measure the angles $x_1 q$ and $x_1 q'$ in opposite directions, each equal to $45°$.

Then the plane (p, q) (and its absolutely perpendicular plane) is the required plane to which the series α and β are isocline in opposite senses at an angle of $45°$. The plane (p, q') also possesses the same property.

Taking the lines x_1, x_2, x_3, x_4 as the axes of an orthogonal frame, p is the line $\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$, and the line q is at right angles to both x_3 and x_2, and makes an angle $45°$ with x_1.

Therefore, its direction-cosines q_i are given by

$$q_1 = \frac{1}{\sqrt{2}}, \quad q_2 = 0, \quad q_3 = 0,$$

and since $\Sigma q_i^2 = 1$, we have

$$q_4 = \frac{1}{\sqrt{2}},$$

so that q is the line

$$\left(\frac{1}{\sqrt{2}}, \ 0, \ 0, \ \frac{1}{\sqrt{2}}\right).$$

The orientation-cosines of the plane (p, q) are, therefore, given by the matrix

$$\left\| \begin{matrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{matrix} \right\|.$$

Hence, if θ_1 and θ_2 are the critical angles between the plane (p, q) and $\alpha(x_1, x_2)$, we obtain, by using the formulae of § 82

$$c_1 \equiv \cos\theta_1.\cos\theta_2 = \tfrac{1}{2}, \text{ and } c_2 \equiv \sin\theta_1.\sin\theta_2 = \tfrac{1}{2},$$

whence $\theta_1 = \theta_2 = 45°$, *i. e.*, the plane (p, q) is isocline to $\alpha(x_1, x_2)$ at an angle of $45°$.

Similarly, the plane (p, q) is found to be isocline to $\beta(x_1, x_3)$ at an angle of $45°$, but in a sense opposite to that in which it is isocline to α (§126).

It now remains to be proved that any plane of either series is isocline to the plane (p, q) at an angle of $45°$.

Take a line S (s_i) in the plane $\alpha(x_1, x_2)$ and a line T(t_i) in $\alpha'(x_3, x_4)$, such that $(x_1 s) = s_1$, $(x_2 s) = s_2$

and $\qquad t_3 = (x_3 t) = -s_1$ and $t_4 = (x_4 t) = s_2$.

Thus the lines are S$(s_1, s_2, 0, 0,)$ and T$(0, 0, -s_1, s_2)$.

Now, if ϕ_1 and ϕ_2 are the critical angles between the planes (S, T) and (p, q), we have

$\cos\phi_1 . \cos\phi_2 = [st/pq]$

$$= \left\| \begin{matrix} s_1 & s_2 & 0 & 0 \\ 0 & 0 & -s_1 & s_2 \end{matrix} \right\| \times \left\| \begin{matrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{matrix} \right\|$$

$$= \tfrac{1}{2} \begin{vmatrix} s_1 & 0 \\ 0 & s_1 \end{vmatrix} + \tfrac{1}{2} \begin{vmatrix} s_2 & 0 \\ 0 & s_2 \end{vmatrix}$$

$$= \tfrac{1}{2}(s_1^2 + s_2^2) = \tfrac{1}{2}.$$

Similarly, $\sin \phi_1 . \sin \phi_2 = [stpq]$

$$= \begin{vmatrix} s_1 & s_2 & 0 & 0 \\ 0 & 0 & -s_1 & s_2 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{vmatrix}$$

$$= \tfrac{1}{2}\,(s_1{}^2 + s_2{}^2) = \tfrac{1}{2}, \qquad (134.1$$

whence $\phi_1 = \phi_2 = 45°$.

Similarly, it can be shown that any plane of the conjugate series $\beta\ (x_1, x_3)$ is inclined to the plane (p, q) at an angle of 45°, but in the opposite sense.

The converse theorem is also true, namely, that *all the planes isocline to a given plane at an angle of 45° lie in two conjugate series.*

135. Theorem: *Non-opposite half-lines p, q and p', q' are respectively taken in two isocline planes α and β (not absolutely perpendicular) such that $\angle pq = \angle p'q'$. Then the planes (p, p') and (q, q') are isocline in the opposite sense.*

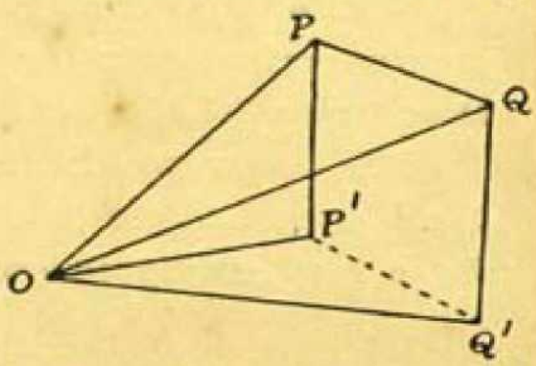
From the mode of construction of isocline planes explained in § 125, it is easily seen that, since the planes α and (p, p') intersect, a plane γ can be constructed (in two ways) to which α and (p, p') are isocline in opposite senses, making the same angle θ. Similarly, for β and the plane (p, p'). Thus, there exists a plane γ to which (p, p') is isocline in one sense, but α and β are isocline in the opposite sense, making, however, the same angle θ.

Consider a plane through q isocline to γ and to (p, p') in a sense opposite to that of β. This plane then

intersects β in a line which makes with (p, p') the same angle that q makes. But the lines q and q' make equal angles with the plane (p, p'), as can easily be seen. Hence, (q, q') is the plane, which is isocline to (p, p') in a sense opposite to that of α and β.*

136. Theorem: *Any plane polygon is similar to its projection on an isocline plane.*

Let PQ be a side of a polygon in the plane POQ and let P′Q′ be its projection on a plane P′OQ′ isocline to POQ, P′ and Q′ being the projections of P and Q respectively.



Since the planes are isocline, $\angle POP' = \angle QOQ'$, and the angles OP′P and OQ′Q are each a right angle.

$\therefore$ The triangles OPP′ and OQQ′ are similar,

so that $OP : OP' = OQ : OQ'$, or, $OP : OQ = OP' : OQ'$

also $\angle POQ = \angle P'OQ'$.

Hence, the triangles POQ and P′OQ′ are also similar, *i.e.*, any triangle is similar to its projection on an isocline plane.

Any polygon can be divided into a number of triangles by joining the vertices to any point in its plane. These triangles are respectively similar to the triangles obtained in the same manner from the projected polygon in the isocline plane. Hence, the polygon

* H. P. Manning, *Geometry of Four Dimensions*, § 111, Th. 2, p. 195.

and its projection on an isocline plane are also similar.

If $\triangle$ represents the area of the polygon and θ be the isoclinal angle, then the area of its projection $= \triangle \cos^2\theta$. In general, if ϕ and ψ are the critical angles between the planes, the area of the projection $= \triangle \cos\phi . \cos\psi$.

The converse theorem is also true, namely, *if a plane polygon is similar to its projection on another plane, the two planes are isocline.*

Cor. : If we consider a regular polygon inscribed in a circle, by increasing the number of sides of the polygon indefinitely, the polygon in its limiting form can be made to approximate the circle. Hence, we deduce that *the projection of a circle upon an isocline plane is also a circle.**

137. Isocline Rotation :

A rotation around the plane $\alpha\,(x,\ y)$ through an angle θ and a simultaneous rotation around the plane $\alpha'(z, w)$ through the same angle θ will convert any point $P(x, y, z, w)$ in the original position to a new position $P'(x', y', z', w')$, by means of the following scheme of transformation† :—

$$\left.\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \\ z' &= z\cos\theta + w\sin\theta \\ w' &= -z\sin\theta + w\cos\theta \end{aligned}\right\} \qquad (137.1$$

* This also follows from what has been stated in § 79.

† See § 115.

If $OP=r=OP'$, the direction-cosines of OP and OP' are respectively

$$\frac{x}{r},\ \frac{y}{r},\ \frac{z}{r},\ \frac{w}{r}, \quad \text{and} \quad \frac{x'}{r},\ \frac{y'}{r},\frac{z'}{r},\frac{w'}{r}.$$

If now ϕ and ϕ' are the critical angles between the planes POP' and (x, y), we have

$$\cos^2\phi.\ \cos^2\phi'.\ \sin^2\theta=\left(\frac{xy'-x'y}{r^2}\right)^2=\frac{(x^2+y^2)^2\sin^2\theta}{r^4}$$

$$\therefore\quad \cos^2\phi.\ \cos^2\phi'=\left(\frac{x^2+y^2}{r^2}\right)^2, \text{ or, } \cos\phi.\ \cos\phi'=\cos^4\alpha,$$

where α is the angle which OP makes with the plane (x, y), showing that ϕ and ϕ' are two equal angles, each equal to the angle which OP or OP' makes with the plane (x, y).

This shows that the line OP rotates in a plane isocline to the axis-plane (x, y), inclined at an angle which OP makes with it.

It is a matter of simple calculation to show that any plane isocline to the plane (x, y) remains isocline to it after rotation, and make the same angle with it.

For, if $lx+my+nz+pw=0$, $l'x+m'y+n'z+p'w=0$ is a plane isocline to the plane $\alpha(x, y)$, the function $lm'-l'm$ remains invariant under the above transformation, but this is proportional to the orientation-cosine corresponding to the plane (x, y).

Hence, all the planes isocline in the sense corresponding to that of the rotation do not change their respective positions, except that they rotate on themselves. Hence it also follows that the series conjugate to any series of isocline planes move as a series on itself.

This double rotation around two absolutely perpendicular planes through the same angle in the same direction may be called an *isocline rotation*, owing to the fact that in this rotation any plane remains isocline to itself (§ 138). The common value of the angle of rotation may be called the *angle of the isocline rotation*.

It is to be noted that any one of the planes of the series and its absolutely perpendicular plane may be taken as the axis-planes of the isocline rotation, *i.e.*, of the constituent rotations.

138. Theorem: *Any plane, after an isocline rotation, remains isocline to itself in the sense opposite to the rotation. If, however, it is isocline to the axis-planes, it rotates on itself.*

Let two lines p and q specify a plane In an isocline rotation, they rotate in two planes isocline to the axis-plane and occupy two new positions p' and q' (say), so that $\angle pp' = \angle qq'$.

Therefore, from what has been said in § 135, it follows that the planes (p, q) and (p', q') are isocline, but in a sense opposite to that in which (p, p') and (q, q') are isocline, *i.e.*, opposite to the rotation.

In case the plane (p, q) is isocline to the axis-plane, the planes (p, q) and (p', q') coincide, *i.e.*, the plane (p, q) rotates on itself.

139. Analogy between the System of isocline Planes in a Fourfold to the Theory of Parallels in an Elliptic Space:

The theory of isocline planes in a fourfold is clearly analogous to Clifford's theory of parallels in elliptic

space of three dimensions with planes as elements. All these planes meet the hypersphere in great circles, which are the straight lines of an elliptic space, whose aggregate is a "space of lines," and absolutely perpendicular planes correspond to the opposite points. The distance between two elements is measured by double the angle between the two planes and the dihedral angle between conjugate planes intersecting in one of the common planes of the two series measures the angle between the two series. An isocline rotation do not produce any motion in this geometry, but a simple rotation or a double rotation corresponds to a rotation of the sphere. The propositions in the system of isocline planes can accordingly be translated into their equivalents in elliptic space by the following dualistic correspondences* :—

In a Fourfold	In elliptic three-dimensional space
Straight lines	Points
Planes	Straight lines (arcs of great circles)
Isocline planes	Parallels
Dihedral angles	Plane angles
Isoclinal angle between two planes.	Perpendicular distance between two straight lines.

* I. Stringham, *Trans. of the Am. Math Soc.*, Vol. 2 (1901), p. 214.

CHAPTER VIII

HYPERSURFACE

140. Curved Loci :

We have so long discussed what has been called *linear spaces*, represented by linear equations in four variables, with the characteristic property of what may be described as *evenness* or *flatness*. Thus, one linear equation represents a hyperplane, which is a linear space of three dimensions, two linear equations represent a plane, a linear space of two dimensions, three linear equations represent a right line, a linear space of one dimension. A suitable representation may similarly be found for equations which are not *all linear*; conversely, non-linear loci may be defined by proper analytical expressions.

When a point, the primary element in a geometric space, moves in any manner continuously in the fourfold, it generates a locus which is called a *curve* or *a line* and is said to be of one dimension. While a curve moves in such a manner that, considering two successive positions, to each point of the first there corresponds one and only one point of the second, and reciprocally, then the curve is said to generate a *surface*, which may be described as of two dimensions. When a surface moves in the fourfold in the same manner, it generates

what may be called a *hypersurface*, and is said to be of three dimensions. The moving point, line and surface may be called the *generating* point, curve and surface respectively, and the curve formed by a series of corresponding points may be called the *guiding curve* or *directrix*. Thus, it appears that a curve may be regarded as the geometric locus of points (of one dimension), a surface as the geometric locus of a curve (of two dimensions), and a hypersurface as the geometric locus of a surface (of three dimensions). This exposition is then found to agree with the earlier statements made in § 14.

141. Analytic Definition:

From what is stated above and from what has already been explained in § 15, a curved loci of three dimensions in the fourfold can now be analytically defined by the postulation of a relation between the variables x, y, z, w, which is *not linear*. This relation imposes one restriction upon the motion of the point, and consequently only three degrees of freedom of motion are left. This definition is analogous to that of a surface in the ordinary space, and is very convenient for discussion of metrical and other topographical properties, rather than projective or intrinsic properties; and for this purpose only those invariantive forms are required which remain unchanged for any orthogonal transformation.

Thus, a single relation between the variables of the form

$$\phi(x, \ y, \ z, \ w)=0 \qquad \dots \quad (1)$$

where ϕ may be of any degree in the variables, represents what may be called a *hypersurface* in the

fourfold. In fact, the locus is of three dimensions, and generally, a locus which is of one dimension less than the containing manifold is called a *hypersurface*. Thus, a conic in a plane is of one dimension, a conicoid in the ordinary space is a surface of two dimensions, and analogously, in the fourfold, a hypersurface is said to have a *curved boundary* of three dimensions.

A hypersurface defined by an equation of order r will be called a *hypersurface of the* rth *order*, and will generally be denoted by V_3^r, where r denotes its order and the suffix 3 indicates that it is of three dimensions.

A hypersurface may also be represented parametrically. Since the co-ordinates are connected by only one equation, three independent parameters are required to express the range of variation of the generating point. Denoting by p, q, r three independent parameters, a hypersurface may be parametrically represented by the equations

$$\left.\begin{array}{ll} x=x\,(p, q, r), & y=y(p, q, r), \\ z=z\,(p, q, r), & w=w(p, q, r) \end{array}\right\} \quad \ldots \quad (2)$$

subject to the condition that a single relation, free from the parameters, exists among the variables.

142. The Order of a Hypersurface :

The ∞^3 aggregate of points in the fourfold, which satisfies a rational, integral equation of the nth order in four variables x, y, z, w, namely,

$$\phi_n(x, y, z, w)=0 \quad \ldots \quad (1)$$

is called a *Hypersurface* of order n and is denoted by V_3^n.

We may write the equation (1) in the extended form

$$
\begin{aligned}
&a \\
&+bx+cy+dz+ew \\
&+fx^2+gy^2+hz^2+iw^2+2jxy+2kyz+2lzx \\
&+2mxw+2nyw+2pzw+ \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&+sx^n+\cdots\cdots\cdots\cdots\cdots\cdots+\text{etc}=0 \quad \ldots \quad (2)
\end{aligned}
$$

or, it may be written symbolically in the form

$$u_0+u_1+u_2+u_3+\ldots\ldots\ldots\ldots+u_n=0,$$

where u_0 is a constant term, and u_r stands for the aggregate of terms of the rth degree. It is now evident that u_0 contains a single term, u_1 contains four terms, u_2 consists of 10 terms, and so on.

The total number of terms in the equation, therefore, is the sum of $(n+1)$ terms of the series 1, 4, 10, 20......... , *i.e.*,

$$N(n)\equiv\frac{(n+1)(n+2)(n+3)(n+4)}{1.\ 2.\ 3.\ 4}={}^{n+4}C_4.$$

Hence, the equation of a hypersurface of order n can be made to satisfy $N(n)-1$ conditions and no more, or in other words, the number of necessary conditions to specify a hypersurface of order n is $N(n)-1$.

The hypersurface (1) is said to be *degenerate* or *non-degenerate*, according as the function ϕ breaks up or not into the product of two or more factors of lower orders.

The geometric significance of the order n of a hypersurface is that it is the number of points in which a right line intersects the hypersurface. In fact, the three equations of a right line along with the equation (1) of the hypersurface form a system of four equations having common solutions, which correspond to the roots of an equation of order n, obtained by eliminating three of the variables between the above four equations. If, however, any line meets the hypersurface in more than n points, it lies wholly in the same, the above equation then reducing to an identity.

Similarly, any plane, defined by two linear equations, meets the hypersurface in a curve of order n, and so on. Hypersurfaces are often termed *Varieties* by the Italian School of Geometers.

143. The Tangent Hyperplane:

If $\alpha, \beta, \gamma, \delta$ be the angles made by any radius vector with the axes of a rectangular frame, the above equation (2) may be put into the form of a polar equation by writing $\rho \cos \alpha$, $\rho \cos \beta$, $\rho \cos \gamma$, $\rho \cos \delta$ for x, y, z, w respectively, when an equation of order n in ρ is obtained, the roots of which will give the distances from the origin of the n points in which the radius vector meets the hypersurface.

If, now, the origin be on the hypersurface, one value of ρ is always zero, and we must have $u_0=0$. A second value will be zero, if the co-efficient of ρ vanishes, *i.e.*, if $\alpha, \beta, \gamma, \delta$ be connected by the relation

$$b \cos \alpha + c \cos \beta + d \cos \gamma + e \cos \delta = 0.$$

Multiplying this by ρ, we obtain

$$bx + cy + dz + ew = 0 \tag{143.1}$$

which expresses the fact that the locus of the radius vector is the *hyperplane* defined by the equation (143.1). This is then the only condition necessary in order that the radius vector should meet the hypersurface in two consecutive points.

Hence we conclude that, *in general, through an assumed point on the hypersurface can be drawn an infinitude of lines, each of which will there meet the hypersurface in two consecutive points, and all such lines lie in one and the same hyperplane, which is called the 'tangent hyperplane' to the hypersurface at the point, and is specified by the equation obtained by equating to zero the linear terms in the equation,* namely, $u_1 = 0$.

The general theory of surfaces as developed in the ordinary space can be extended to the hypersurface in the fourfold;* but the curvatures and other intrinsic properties are capable of being studied far more conveniently by means of parametric representation, which provides a very effective mode of treatment of the differential geometry of hypersurfaces.†

144. Quadric Variety :

A hypersurface of the second order is called a *Quadric*, a *Quadric Hypersurface*, or a *Quadric Variety*, and is generally denoted by V_3^2.

* E. Bertini, *Introduzioni alla Geometria, etc.* (1907), Cap. 8 and 9.

† Salmon, *Geometry of Three Dimensions*, 5th ed. (1912), Ch. XI, pp. 268-319.

The most general equation of a Quadric Variety may be written as

$$\phi(x, y, z, w) \equiv ax^2 + by^2 + cz^2 + dw^2$$
$$+ 2fyz + 2gzx + 2hxy + 2lxw + 2myw + 2nzw$$
$$+ 2px + 2qy + 2rz + 2sw + t = 0 \quad \ldots \quad (1)$$

which may be symbolically written in the form

$$\phi(x, y, z, w) \equiv \Sigma ax^2 + 2\Sigma fyz + 2\Sigma px + t = 0$$
$$\equiv (\, a \between x \,) = 0 \quad \ldots \quad (2)$$

The above equation contains 14 independent parameters, and consequently through any fourteen independent points can be drawn one, and only one, Quadric Variety.

The equation of the tangent hyperplane may be obtained by transferring the origin to any point on the Quadric, and then equating the linear terms of the new equation to zero.

If x', y', z', w' be the co-ordinates of any point on the Quadric, the equation of the tangent hyperplane is obtained, as in the geometry of three dimensions,* as follows:

If ϕ_1, ϕ_2, ϕ_3, ϕ_4, denote the differential co-efficients of ϕ with respect to x, y, z and w respectively, and ϕ' denotes the result of substituting x', y', z', w', respectively for x, y, z, w in ϕ, then the equation of the quadric, referred to parallel axes through (x', y', z', w') takes the form

$$\Sigma ax^2 + 2\Sigma fyz + x.\frac{\partial \phi'}{\partial x'} + y\frac{\partial \phi'}{\partial y'} + z\frac{\partial \phi'}{\partial z'} + w\frac{\partial \phi'}{\partial w'} + \phi' = 0 \quad \ldots \quad (3)$$

$$\text{or, } \Sigma ax^2 + 2\Sigma fyz + x\phi_1' + y\phi_2' + z\phi_3' + w\phi_4' + \phi' = 0 \quad \ldots \quad (4)$$

* Salmon, *loc. cit.*, § 62.

But, if (x', y', z', w') is a point on the quadric, $\phi'=0$, and the equation of the tangent hyperplane is obtained by equating to zero the linear terms in the transformed equation, *i.e.*,

$$x\phi'_1+y\phi'_2+z\phi'_3+w\phi'_4=0$$

or, transforming back to the original system of axes,

$$(x-x')\phi'_1+(y-y')\phi'_2+(z-z')\phi'_3+(w-w')\phi'_4=0 \quad (144.1$$

or, writing out in full length, it becomes

$$\begin{aligned} &x(ax'+hy'+gz'+lw'+p)\\ &\quad+y(hx'+by'+fz'+mw'+q)\\ &\quad\quad+z(gx'+fy'+cz'+nw'+r)\\ &\quad\quad\quad+w(lx'+my'+nz'+dw'+s)\\ &\quad\quad\quad\quad+(px'+qy'+rz'+sw'+t)=0 \quad (144.2\end{aligned}$$

This may be written in a more symmetrical form by the introduction of a linear unit u, when, since ϕ is now a homogeneous function, and $\phi'=0$, we have

$$x'\phi'_1+y'\phi'_2+z'\phi'_3+w'\phi'_4+u'\phi'^5\equiv\phi'=0.$$

Adding this to the above equation, we obtain the equation of the tangent hyperplane in the form

$$x\phi'_1+y\phi'_2+z\phi'_3+w\phi'_4+u\phi'_5=0. \quad (144.3$$

The normal to the tangent hyperplane at its point of contact is called the *normal* to the quadric at that point.

It is to be noted that all the lines and planes passing through the point and lying in the tangent hyperplane touch the quadric at the point.

145. Canonical Form of a Quadric Variety :

Transferring the origin to a point such that all radii drawn through it are bisected at the point, the linear terms in the equation (1) can be made to vanish, and then an orthogonal transformation may be so applied that the terms containing xy, yz, zx, etc., all vanish. Thus, the equation of a quadric variety can be transformed into the simplest form

$$Ax^2+By^2+Cz^2+Dw^2=1 \tag{145.1}$$

Let $\phi(x, y, z, w)=0$ be the equation of the Quadric, and (ξ, η, ζ, ρ) be the co-ordinates of the centre. Let X, Y, Z, W be the new variables, referred to a rectangular system through the new origin and t a unit symbol by which the given function ϕ may be made homogeneous.

If, then, ϕ is transformed to ϑ, when t, ξ, η, ζ, ρ are substituted for 1, x, y, z, w respectively, and

$$\mathrm{D}\equiv X\frac{\partial}{\partial\xi}+Y\frac{\partial}{\partial\eta}+Z\frac{\partial}{\partial\zeta}+W\frac{\partial}{\partial\rho}$$

then the transformed equation takes the form

$$\vartheta+\mathrm{D}\vartheta+\tfrac{1}{2}\mathrm{D}^2\vartheta=0 \tag{145.2}$$

If, then, $\mathrm{D}\vartheta=0$, independent of the values of the new variables, the linear terms will be absent from the equation. The necessary conditions are then

$$\frac{\partial\vartheta}{\partial\xi}=0,\ \ \frac{\partial\vartheta}{\partial\eta}=0,\ \ \frac{\partial\vartheta}{\partial\zeta}=0,\ \ \frac{\partial\vartheta}{\partial\rho}=0. \tag{145.3}$$

These equations will then determine the co-ordinates (ξ, η, ζ, ρ) of the centre.

Since $$\vartheta=\tfrac{1}{2}\left(t\frac{\partial\vartheta}{\partial t}+\xi\frac{\partial\vartheta}{\partial\xi}+\eta\frac{\partial\vartheta}{\partial\eta}+\zeta\frac{\partial\vartheta}{\partial\zeta}+\rho\frac{\partial\vartheta}{\partial\rho}\right)$$

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the new equation reduces to

$$\tfrac{1}{2}.\frac{\partial \vartheta}{\partial t}+\tfrac{1}{2}D^2\vartheta=0 \tag{145.4}$$

Hence, if X, Y, Z, W is a point, $-X, -Y, -Z, -W$ is also a point on the quadric and any radius drawn through (ξ, η, ζ, ρ) is bisected there. The point (ξ, η, ζ, ρ) is called the *centre* of the Quadric. The equation is thus reduced to the form

$$f(X, Y, Z, W)=1,$$

where f is a homogeneous function of the second degree in four variables. If now we apply an orthogonal transformation, which involves six independent parameters, these parameters may be so chosen that six of the co-efficients, namely, those of XY, YZ, ZX, XW, YW, ZW vanish, and now only the square terms remain in the equation, so that it takes the form (145.1).

Geometrically, we may rotate the two axes in each of the six co-ordinate planes through a suitable angle in such a manner that the co-efficient of the product of the corresponding variables will be zero.*

Thus, the equation of a Quadric Variety can be reduced to the standard form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}+\frac{w^2}{d^2}=1. \tag{145.5}$$

The new system of axes may be called the *axes* of the quadric, and $2a$, $2b$, $2c$, $2d$ are the lengths of these axes intercepted by it. Just as in the ordinary space, these Quadrics may now be classified into different species of hypersurfaces.

* Salmon, *Conics*, § 156.

The Quadric contains a ∞^3 of lines and through each point, there pass an ∞ of these lines. They form an ordinary cone of the second order, which is contained in the tangent hyperplane at the point. A plane which cuts the quadric in two tangent lines through the point is a *tangent plane*.

If at each point of a line 'l' of the quadric, a tangent hyperplane be drawn, the number of such hyperplanes is ∞^1, and they form a pencil of hyperplanes, having a common face, which touches the quadric at all points of the line l, *i.e.*, this plane meets the quadric in the line l taken twice. The plane may be called a *hypertangent plane*. Through each line of a quadric there passes one hypertangent plane, and consequently their number is ∞^3. The locus of all these hypertangent planes passing through any given point is a hypercone of the first species.*

146. Intersection of Quadric Varieties :

The intersection of two Quadric Varieties represented by the two equations

$$\phi\,(x, y, z, w)=0 \quad \text{and} \quad \psi\,(x, y, z, w)=0,$$

where ϕ and ψ are polynomials of the second order in the variables, is generally a *quartic curve*,† which may be a more general locus of the second degree, analogous to that studied in the geometry of three dimensions.

* To be explained later.

† The subject of hyper-quartics was studied by M. Bordiga, *Studio generale della quartica normale*—Atti dell' Ist. ven. (1886).

In fact, this is analogous to the intersection of two surfaces in the ordinary space, which is a skew curve. Just as the skew curve does not represent the complete intersection of the two surfaces, here also the complete intersection consists of two or more portions which are not geometrically continuous with one another. Generally, however, the above two equations are taken to represent a quadric surface in the fourfold, although other modes of representation for a surface are more convenient and will be discussed subsequently in their proper places.

The equation $\phi+\lambda\psi=0$ represents a pencil of quadric varieties, passing through the common points of the two quadrics, namely, the above skew quartic, which contains five* *hypercones* of the first species.

The three equations $\phi=0$, $\psi=0$ and $\chi=0$, where ϕ, ψ, χ are polynomials of the second degree in the variables, represent a skew curve of the second degree, generally of *triple* curvature.†

147. Homogeneous Co-ordinates :

The Cartesian co-ordinates thus far used are essential in studying intrinsic metrical properties of spaces, curves, etc. As in the case of lower geometries, introduction of homogeneous co-ordinates in the geometry of the fourfold greatly facilitates the study of projective and other descriptive properties of curves,

* See § 159.

† An important work on Hyperquadrics is a Memoir of C. Segre, published in the *Recueil de l'Academie de Turin*. Segre— *Studio sulle quadriche in un spazio lineare un numero qualunque di dimensioni* (Mem. de Turin (2), p. 36, 1884).

surfaces and hypersurfaces.* At this stage, therefore, use of some forms of homogeneous systems of co-ordinates seems desirable and will be much helpful in studying the general nature of hypersurfaces, etc., in the fourfold, where, owing to the lack of general intuitive conception, the matter becomes much more complicated and in many respects incomprehensible.

With Bertini, we may take any five independent points A_0, A_1, A_2, A_3, A_4 to form the vertices of a *simplex* (a pentahedroid)† which has for boundary the five tetrahedra formed by each group of four of the five points. The five vertices may be called the *fundamental points*, or *points of reference*, and the simplex may be called the *simplex of reference*, or the *fundamental simplex*.

In fact, the simplex is bounded by the five hyperplanes determined by each group of four of the fundamental points. Special systems of homogeneous co-ordinates, corresponding to the trilinear or areal systems in planes can now be defined.

The five hyperplanes, defined respectively by the five linear equations

$$A_0=0, \quad A_1=0, \quad A_2=0, \quad A_3=0, \quad A_4=0,$$

may be taken as five hyperplanes of reference. The position of a point P, referred to these hyperplanes, may be specified by means of quantities proportional to the lengths of the perpendiculars drawn from P on these hyperplanes. This is analogous to the trilinear system of co-ordinates.

* A comprehensive study of the projective properties of hyperspaces has been made by E. Bertini, *Introduzioni alla geometria proiettiva degli iperspazi*, etc., Pisa (1907).

† To be explained later.

The point U (1, 1, 1, 1, 1) plays a very important part in this system of representation and is called a *unit point*.

The co-ordinates of the five fundamental points are A_0 $(a, 0, 0, 0, 0)$, A_1 $(0, b, 0, 0, 0)$, A_2 $(0, 0, c, 0, 0)$, A_3 $(0, 0, 0, d, 0)$, A_4 $(0, 0, 0, 0, e)$, where a, b, c, d, e are the lengths of the perpendiculars drawn from the fundamental points on the opposite hyperplanes.

With the help of the unit point, the co-ordinates of any point P can now be defined in another manner.

Let $(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$ be the co-ordinates of P. Join UP and produce it to meet the hyperplane opposite A_i in P_i, *i.e.*, let the line UP meet the five hyperplanes respectively in the points

$$P_0, P_1, P_2, P_3, P_4.$$

The co-ordinates of any point on UP may be taken as

$$\rho x_i = \lambda \alpha_i + \mu.1, \ (i=0, 1, 2, 3, 4)$$

where λ and μ are two parameters.

Different points on the line UP are obtained by different values of the ratio of the homogeneous parameters

$$\lambda_i / \mu_i \equiv p_i \quad (i=1, 2, 3, 4).$$

Let p_0, p_1, p_2, p_3 denote these ratios for the four points P_o, P_i, U, P respectively.

Now, the parameters of the four points P_0, P_i, U, P are respectively given by

$$0=\lambda_0\alpha_0+\mu_0, \quad 0=\lambda_1\alpha_i+\mu_1, \quad \lambda_2=0, \quad \mu_3=0$$

The cross-ratio

$$(\mathrm{P}_0\mathrm{P}_i,\ \mathrm{UP})=(p_0p_1,\ p_2p_3)=\frac{(p_0-p_2)(p_1-p_3)}{(p_1-p_2)(p_0-p_3)}$$

$$=\frac{-1/\alpha_0}{-1/\alpha_i}=\frac{\alpha_i}{\alpha_0}.$$

Denoting this ratio by X_i, we get

$$\mathrm{X}_i=\frac{\alpha_i}{\alpha_0}\quad (i=1,\ 2,\ 3,\ 4) \tag{147.1}$$

i.e., the ratios of the four co-ordinates to α_0 are expressed as four cross-ratios. Thus, to determine the co-ordinates of any point P, we join it to the unit point U and produce UP to meet the hyperplanes of reference in five points

$$\mathrm{P}_0,\ \mathrm{P}_1,\ \mathrm{P}_2,\ \mathrm{P}_3,\ \mathrm{P}_4.$$

Then, the cross-ratio which the three points P_0, U, P form with each of the remaining four points will determine the ratios of the corresponding co-ordinates to α_0.

If, then, the distances of the points U and P from the fundamental hyperplanes be denoted by u_i and l_i respectively, we have

$$\frac{\alpha_i}{\alpha_0}=(\mathrm{P}_0\mathrm{P}_i,\ \mathrm{UP})=\frac{\mathrm{P}_0\mathrm{U}}{\mathrm{P}_0\mathrm{P}}\div\frac{\mathrm{P}_i\mathrm{U}}{\mathrm{P}_i\mathrm{P}}=\frac{u_0}{l_0}\div\frac{u_i}{l_i} \tag{147.2}$$

whence $\alpha_i=k.l_i/u_i$, where k is constant, showing that the co-ordinates α_i are multiples of the lengths of perpendiculars drawn from the point P to the fundamental hyperplanes.

The co-ordinates α_i $(i=0,\ 1,\ 2,\ 3,\ 4)$ are connected by an identical relation, just as the trilinear co-

ordinates in a plane are. Joining the point P to the vertices of the simplex, it is divided into five simplexes.

If now V_i denote the content of the simplex (tetrahedron) opposite to the vertex A_i, the content V of the simplex with base V_i and vertex P

$$=\tfrac{1}{4}\, V_i\, l_i$$

$$\therefore\ \Sigma V_i l_i = 4V \quad \text{and} \quad \Sigma V_i u_i \alpha_i = \Sigma V_i k l_i = 4kV \qquad (147.3$$

i.e., the co-ordinates are connected linearly by an identity. If, however,

$$\Sigma V_i u_i \alpha_i = 0,$$

since the equation is linear and homogeneous, it represents a hyperplane. But in virtue of the relation (147·3), no finite values of the co-ordinates will satisfy this equation, showing that the locus does not contain any finite point. Hence it represents the hyperplane at infinity (§ 46).

Note. All equations in this system can be made homogeneous in virtue of the relation (147·3).

148. A second System of Homogeneous Co-ordinates:

In this system, the fundamental simplex is divided into five simplexes by joining the point P to the fundamental points. The co-ordinates of any point P may be defined as the ratios of the *contents* of the five simplexes to that of the fundamental simplex.

Denoting the fundamental points, as before, by A_0, A_1, A_2, A_3, A_4, and the co-ordinates of P by $x_i\,(i=0, 1, 2, 3, 4)$, let V_i denote the content of the simplex with P as the vertex and the tetrahedron opposite A_i as the base.

Then, $$x_i = V_i / V, \qquad \ldots \quad (148.1$$

where V denotes the content of the fundamental simplex.

These are analogous to the areal or four-plane co-ordinate systems in the lower geometries of two and three dimensions respectively, and may be called the *contental system* of co-ordinates.

By definition, we have

$$\Sigma x_i = \Sigma \mathrm{V}_i / \mathrm{V}, \ (i=0, 1, 2, 3, 4)$$

$$i.e., \quad x_0 + x_1 + x_2 + x_3 + x_4 = \frac{1}{\mathrm{V}}(\mathrm{V}_0 + \mathrm{V}_1 + \mathrm{V}_2 + \mathrm{V}_3 + \mathrm{V}_4)$$

$$= 1, \qquad (148.2$$

showing that the sum of the co-ordinates is equal to unity, or in other words, they are connected by the identical relation (148.2). This relation is always true, wherever the point P is taken in the finite part of the fourfold; when it lies outside the simplex, proper regard must be had to the signs of the contents. For a point inside the simplex, the signs are all positive; for a point lying outside, the sign is to be determined from the fact that it is regarded as positive, if the single perpendicular, drawn from the given point on any of the simplexes, is in the same direction as the perpendicular drawn from the corresponding fundamental point, and negative in the opposite case.

Note 1. Notice that none of these co-ordinates can be infinite, and all of them cannot simultaneously vanish. In virtue of the identical relation (148.2), all equations in this system can be made homogeneous. The equation of the hyperplane at infinity is $x_0 + x_1 + x_2 + x_3 + x_4 = 0$.

Note 2. Both these systems of homogeneous co-ordinates can be transformed into the usual cartesian system by expressing the contents of the simplexes in terms of the cartesian co-ordinates of the fundamental points. (See §§ 29 and 30.)

149. The Cartesian as a special system of homogeneous Co-ordinates:

Just as in the plane geometry, the Cartesian system may be regarded as a special system of homogeneous co-ordinates by considering the hyperplane at infinity as one of the fundamental hyperplanes, say $x_0=0$. The opposite vertex is now called the origin O, the fundamental lines through it are then the co-ordinate axes, and the fundamental hyperplanes through O are the co-ordinate hyperplanes. The unit point is now taken as the point whose distance from each of the co-ordinate hyperplanes, measured parallel to the opposite axis, is unity. Thus, the homogeneous co-ordinates x_0, x_1, x_2, x_3, x_4 may be regarded as a system of cartesian co-ordinates, in which $x_0=0$ is the hyperplane at infinity, *i.e.*, we may convert a homogeneous equation into a cartesian form by replacing x_0 by a constant,* say unity; and conversely, any cartesian equation may be made homogeneous by introducing requisite powers of x_0 in the different terms, where $x_0=1$. These homogeneous systems can, therefore, be conveniently used for studying projective properties of hypersurfaces, and in particular, Quadric Varieties.

150. Special Forms of Equation of a Quadric Variety:

A Quadric Variety may be represented by either of the two forms of equations

$$\sum_0^4 a_i x_i^2 + 2\Sigma a_{ij} x_i x_j = 0 \qquad (150.1$$

$$(i=j=0,\ 1,\ 2,\ 3,\ 4)$$

* See § 46.

or $$\sum_{i=o}^{4} \sum_{j=o}^{4} a_{ij} x_i x_j = 0 \tag{150.2}$$

where $a_{ij} \equiv a_{ji}$.

For brevity of expressions, we may write either of these equations in the form

$$(a \between x)^2 = 0, \quad \text{or,} \quad f(x_0, x_1, x_2, x_3, x_4) = 0. \tag{150.3}$$

151. Conjugate Points :

Let P and Q be any two points whose co-ordinates are α_i and β_i respectively. If then R is a point on PQ, the co-ordinates of R may be taken as

$$\rho x_i = \lambda \alpha_i + \mu \beta_i \quad (i = 0,\ 1,\ 2,\ 3,\ 4).$$

If R lies on the Quadric, these co-ordinates must satisfy its equation, and we must have

$$\rho^2 (a \between x)^2 = \lambda^2 (a \between \alpha)^2 + 2\lambda\mu (a \between \alpha \between \beta)$$
$$+ \mu^2 (a \between \beta)^2 = 0$$

where $(a \between \alpha \between \beta) \equiv \Sigma a_i \alpha_i \beta_i + \Sigma a_{ij} (\alpha_i \beta_j + \alpha_j \beta_i)$.

This is a quadratic equation in $\lambda : \mu$, and therefore determines the two points in which the line PQ meets the Quadric. If, however, the equation (1) is identically satisfied, the line PQ meets it in more than two points, and consequently lies entirely on the Quadric.

If the equation has a double root, then the two points in which PQ meets the Quadric are coincident, and in this case, the line PQ touches the Quadric, or is a *tangent line*, the point of contact corresponding to the double root.

If P lies on the Quadric, one root is zero, and this requires that $\mu=0$, and we get $(a \between \alpha)^2=0$.

If another root is to be zero, *i.e.*, if Q coincides with P on the Quadric, then $(a \between \alpha \between \beta)=0$. If now P is fixed and Q varies, by making β current, the locus of any point Q on the line PQ, which meets the Quadric in two coincident points at P, is obtained in the form

$$(a \between \alpha \between x)=0. \tag{151.1}$$

This is a hyperplane and we obtain the theorem that *the locus of all lines which touch a Quadric at a point is a hyperplane, which is called the tangent hyperplane at the point.*

If β be fixed and α variable, the locus of points of contact of all tangent lines drawn through β lie on the hyperplane $(a \between \beta \between x)=0$.

Let A and B be the two points in which PQ meets the Quadric.

If, then, A and B are harmonic conjugates of P and Q, the two roots of (1) are equal but opposite in sign, the condition for this is that

$$(a \between \alpha \between \beta)=0. \tag{151.2}$$

The two points P and Q are then said to be *conjugate* with regard to the Quadric. In this case the equation (1) reduces to

$$\lambda^2(a \between \alpha)^2+\mu^2(a \between \beta)^2=0.$$

The two points of intersection are therefore of the forms $\qquad \lambda\alpha_i \pm \mu\beta_i$.

Hence, the points P, Q and A, B form a harmonic range. If now, P remains fixed, the locus of Q, the harmonic conjugate of P with regard to A, B, *i.e.*, the

polar of P, is obtained by making β current, *i.e.*, the polar of the point $P(\alpha)$ is given by $(a \between \alpha \between x) = 0$. (151.3

This is a hyperplane and is called the *polar hyperplane* of the point α. It will be seen that this, as stated above, is the locus of points of contact of all tangent lines drawn from the point α. The point α is called the *pole*.

When, however, the point α lies on the Quadric, the polar hyperplane becomes the tangent hyperplane at α, *i.e.*, *the polar hyperplane of a point on the Quadric is the tangent hyperplane at the point.*

In this case, the points on the Quadric may be said to be *reciprocally polar to themselves, or self-polar with regard to the Quadric.*

This may be extended to higher dimensions as well, and two spaces are said to be *conjugate*, when the polar hyperplanes of points in one have the other common between them.*

Ex. 1. If P lies on the polar hyperplane of Q, Q lies on the polar hyperplane of P.

Ex. 2. If P moves in a hyperplane, its polar hyperplanes all pass through a common point Q, which is the pole of the locus of P (a hyperplane).

Ex. 3. The polar hyperplanes of two points P and Q form a pencil. The common face then corresponds to the line PQ. Reciprocally, if the point P moves in a plane, its polar hyperplanes all pass through a common line, determined by the poles of any two hyperplanes of the pencil.

* Veronese, *loc. cit.*, § 159, p. 571. The general theory of *polarisation in the geometry of n dimensions* has been discussed analytically by E Bertini, *Introduzione alla geometria proiettiva degli Iperspazi*, etc., Cap 8°, pp. 164-88.

Note. It is now easily seen that points, lines, planes and hyperplanes can be connected by a reciprocal relation, *i.e.*, that of poles and polars of the above nature.

A hyperplane (determined by 4 points) has one pole (determined by four hyperplanes).

A plane (determined by 3 points) has two poles (points on a line, determined by 3 hyperplanes).

A line (determined by two points) has three poles in a plane (determined by 2 hyperplanes).

A point has four poles (lying in one hyperplane).

Hence, the geometrical entities in the fourfold can be associated by a dual relation, the two being determined by the same number of points and the same number of hyperplanes respectively.

Thus, a hyperplane may be said to be *conjugate* to a point, a plane to a line, a point to a hyperplane.

152. Quadric Variety as an Envelope :

The condition that any given hyperplane may be a tangent hyperplane to a Quadric, *i.e.*, it may touch the Quadric at any point is obtained by identifying the equation of the given hyperplane and that of the tangent hyperplane at the point.

The hyperplane $\Sigma\alpha_i x_i=0$ will, therefore, touch the Quadric $(a \between x)^2=0$, if

$$\begin{vmatrix} a_{00} & a_{01} & a_{02} & a_{03} & a_{04} & \alpha_0 \\ a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & \alpha_1 \\ a_{20} & a_{21} & a_{22} & a_{23} & a_{24} & \alpha_2 \\ a_{30} & a_{31} & a_{32} & a_{33} & a_{34} & \alpha_3 \\ a_{40} & a_{41} & a_{42} & a_{43} & a_{44} & \alpha_4 \\ \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & 0 \end{vmatrix}=0. \quad (152.1$$

This may be written in the form

$$\sum_{0}^{4} \sum_{0}^{4} A_{ij}\alpha_i\alpha_j=0, \quad \text{or,} \quad (A \between \alpha)^2=0, \qquad (152.2$$

where $A_{ij} \equiv \dfrac{\delta\Theta}{\delta a_{ij}}$, Θ being the leading minor of order 5 of the determinant (152.1), *i.e.*,

$$\Theta \equiv | \, a_{00}, a_{11}, a_{22}, a_{33}, a_{44} \, | \, .$$

If, then, the co-efficients in the equation of the given hyperplane be denoted by λ_i, the condition (152.1) may be put into the form $(A \between \lambda)^2=0$. (152.3

Thus, we see that the point x_i lies on the quadric variety $(a \between x)^2=0$, and the hyperplane with parameters λ_i is the tangent hyperplane to the envelope $(A \between \lambda)^2=0$.

The equation $(A \between \lambda)^2=0$ may be called the *tangential polar equation* of the quadric. Hence, a quadric variety may be regarded as the *locus* of a point defined by the equation,

$$(a \between x)^2=0$$

and as the *envelope* of a hyperplane defined by the equation

$$(A \between \lambda)^2=0.^* \qquad (152.4$$

The two equations may be called *reciprocal* to each other.

153. Self-Polar Simplex:

The simplex of reference may be so chosen that the fundamental points may be *reciprocally polar* with regard to a given quadric.

* Cayley, *Phil. Trans. of R. Soc. of London*, Vol. CLX (1870).

If $(a \between x)^2 = 0$ is a given quadric variety, the polar hyperplane of the fundamental point $A_0(1, 0, 0, 0, 0)$ with regard to this may be written as

$$(a \between x \between 1) = 0.$$

But this must be the fundamental hyperplane $x_0 = 0$, which requires that

$$a_{01} = a_{02} = a_{03} = a_{04} = 0.$$

Similarly for other fundamental points A_1, A_2, A_3, A_4.

The equation of the quadric is now reduced to the form :—

$$a_{00}x_0^2 + a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + a_{44}x_4^2 = 0 \quad (153.1$$

and the fundamental simplex is called a *self-polar* simplex with regard to this quadric.

A self-polar simplex may be constructed in the following manner :—

Take a point A_0 not on the quadric. The polar hyperplane of A_0 does not contain A_0. The intersection of this hyperplane with the given quadric variety is a quadric of 2 dimensions, *i.e.*, a conicoid. * Take any point A_1 in this polar hyperplane, but not on the conicoid. Again, take a third point A_2 in the plane of intersection of the polar hyperplanes of A_0 and A_1, and a fourth point A_3 on the intersection of the three polar hyperplanes of A_0, A_1, A_2. The polar hyperplanes of the four points A_0, A_1, A_2, A_3 will intersect in a point A_4, which is to be taken as the fifth point. Thus, five independent points are obtained, all reciprocally polar to each other with regard to the given quadric variety.

* Bertini, *loc. cit.*, Cap. VIII, Nos. 2 and 3.

154. Generating Manifolds :

In the geometry of the ordinary space, through any point on a quadric, two generating lines, real or imaginary, can be drawn. This can be generalised as follows* :—

If the complete manifold be of 2μ or $2\mu-1$ dimensions, the subordinate manifolds, real or imaginary, contained in any quadric variety will be of μ—1 dimensions.

The general theorem for an n-dimensional space can be established by taking two points α_1 and α_2 mutually reciprocally polar and on the quadric. Their polar hyperplanes are each of order $(n-1)$ and intersect in a hyperplane of order $(n-2)$, which intersects the quadric in a quadric of $(n-2)$ dimensions. Take a third point α_3 on this quadric. Then α_1, α_2, α_3 are reciprocally polar to each other, and any point in their plane lies in the intersection of the three polar spaces and on the quadric. This latter locus is a space of $(n-3)$ dimensions, and taking a fourth point α_4, we may proceed as before. Suppose k points, mutually reciprocal, have thus been selected on the quadric. If they are linearly independent, their polar spaces intersect in a space of $(n-k)$ dimensions containing these k points.

$\therefore$ We must have

$$n-k+1 \geq k$$

i.e., $2k \leq (n+1)$. Or, the greatest value of k is the highest integer in $\frac{n+1}{2}$. If then $n=2\mu$, or $2\mu-1$, the value of k must be μ, and these μ independent

* Veronese, *loc. cit.*, Buch II, Sec. 15.

points determine a space of $(\mu-1)$ dimensions, contained on the quadric, *i.e.*, the real or imaginary generating manifold of the quadric is a manifold of $(\mu-1)$ dimensions.

In the fourfold, $\mu=2$, and we have only *generating lines* of a quadric variety.

In this case, we have to determine how many reciprocally polar and independent points can be taken on the quadric. These points will determine a linear manifold of proper dimensions lying entirely on the quadric.

Take two points α_1 and α_2 on the quadric reciprocally polar to each other. We have, then, $(a\between\alpha_1\between\alpha_2)=0$. Any point on the line $\alpha_1\alpha_2$, therefore, lies in both the polar hyperplanes and also on the quadric. The two polar hyperplanes intersect in a plane, which again passes through this line and is the hypertangent plane (§ 145).

The line $\alpha_1\alpha_2$, therefore, lies entirely on the quadric. If α_3 is a third point on the quadric, conjugate to both α_1 and α_2, but not lying on the line $\alpha_1\alpha_2$, then every point of the plane $\alpha_1\alpha_2\alpha_3$ will lie entirely on the quadric. Since the hypertangent plane passes through the line $\alpha_1\alpha_2$, no third point α_3 can be taken as above.

Hence, only two independent points can be taken on the quadric, which are reciprocally polar to each other, or in other words, a quadric variety in the fourfold has only *generating lines*, real or imaginary.

Cor.: Each generating line is defined by two points, and consequently, from any point only one straight line can be drawn, intersecting two non-inter-

secting generating lines. If, however, the point be on the quadric, and does not lie on any of the generating lines, the line meets the quadric in three points and therefore lies wholly on the quadric.

Thus, *from any point on a Quadric Variety, one and only one line can be drawn meeting any two non-intersecting generating lines, and this itself is a generating line.*

155. Closed Quadrics :

A Quadric is said to be *closed*, if points, not on the quadric, exist such that all right lines drawn through any of them cut the quadric each in two real points. If such points exist, they are said to be *within* the quadric. Other points, not lying on the quadric and not possessing this property, are said to be *without* the quadric. Thus, a closed quadric divides the points of the fourfold in three groups, namely, (1) those lying within, (2) those lying without, and (3) those lying on the quadric.

Let any straight line, drawn through a point P (α_i), meet the quadric $(a \between x)^2=0$ in two points $A(y_i)$ and $B(z_i)$, real or imaginary.

If x_i be any other real point on this line, then y_i and z_i will be of the forms

$$y_i=\lambda_1\alpha_i+\mu_1 x_i, \qquad z_i=\lambda_2\alpha_i+\mu_2 x_i$$

where λ_1/μ_1 and λ_2/μ_2 are the roots of the equation

$$\lambda^2(a \between \alpha)^2+2\lambda\mu(a \between \alpha \between x)+\mu^2(a \between x)^2=0. \quad \dots (1)$$

The roots of this equation will be real or imaginary, according as

$$(a \between x)^2.(a \between \alpha)^2-\{(a \between \alpha \between x)\}^2 \qquad \dots (2)$$

is negative or positive.

If, now, we choose a self-polar simplex for reference, with one vertex at P, the co-ordinates x_i can be taken as

$$x_i = \xi_0 \alpha_i + \xi_1 \beta_i + \xi_2 \gamma_i + \xi_3 \delta_i + \xi_4 \rho_i$$

where β_i, γ_i, δ_i, ρ_i are the remaining vertices of the simplex, and ξ_0, ξ_1, ξ_2, ξ_3, ξ_4, are parameters.

Then, from what has been stated in §§ 147 and 153, $(a \between x)^2$ takes the form

$$a_0 x_0^2 + a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + a_4 x_4^2$$

and the condition (2) now takes the form

$$a_0(a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + a_4 x_4^2) \gtreqless 0. \qquad (155.1)$$

When the points of intersection A, B are imaginary, the expression (2) is positive for all values of x_1, x_2, x_3, x_4.

Hence, $a_0\, a_1$, $a_0\, a_2$, $a_0\, a_3$, $a_0\, a_4$ must be all positive, *i. e.*, a_0, a_1, a_2, a_3, a_4 must be all of the same sign.

The quadric is then a *virtual* quadric, and its equation now takes the form—

$$p_0^2 x_0^2 + p_1^2 x_1^2 + p_2^2 x_2^2 + p_3^2 x_3^2 + p_4^2 x_4^2 = 0. \qquad (155.2)$$

If, however, the points A, B are real, then $a_0 a_1$, $a_0 a_2$, $a_0 a_3$, $a_0 a_4$ must all be negative.

Hence, the sign of a_0 must be different from that of a_1, a_2, a_3, a_4, which are to be all of the same sign. The equation of the quadric will then take the form—

$$p_1^2 x_1^2 + p_2^2 x_2^2 + p_3^2 x_3^2 + p_4^2 x_4^2 - p_0^2 x_0^2 = 0. \qquad (155.3)$$

This shows that the point $P(x_0)$ is *within* the quadric and the remaining four vertices of the simplex are *without* the quadric on the polar hyperplane of P.

Cor.: It follows immediately that the polar hyperplane $(x_0=0)$ of a point P inside the quadric does not intersect the quadric in real points.

For the intersection is the imaginary quadric

$$p_1{}^2x_1{}^2+p_2{}^2x_2{}^2+p_3{}^2x_3{}^2+p_4{}^2x_4{}^2=0.$$

Similarly, from equation (155.3) it follows that the polar $(x_1=0$, say) of any other point intersects the quadric in the coincoid

$$p_2{}^2x_2{}^2+p_3{}^2x_3{}^2+p_4{}^2x_4{}^2-p_0{}^2x_0{}^2=0$$

which is real. Hence, *the polar hyperplane of any point without a closed quadric intersects the quadric in real points and contains points within the quadric.*

It is to be noted that no real generating lines exist on a closed quadric.

156. Conical Quadric Variety:

Corresponding to cones in the geometry of the ordinary space, we have special Quadric Hypersurfaces in the fourfold, which may be called *Conical Quadric Hypersurfaces*, or *Quadric Hypercones.*

After Veronese,* we may define a Hypercone as follows:—

If V is a quadric in a hyperplane, and O a point not lying in its hyperplane, the ruled hypersurface, generated by the straight lines joining O to the points of the quadric V, is a *hypercone*. V is called the *base* or *directing quadric* and O the *vertex*, and the generating lines are called the *elements*.

* Veronese, *loc. cit.*, § 179, Def. 1.

Let A_0 (1,0,0,0,0) be the vertex and the base be defined by the equations

$$\Sigma\Sigma a_{ij} x_i x_j = 0, \; (i=j=0, 1, 2, 3, 4) \text{ and } \Sigma \alpha_i x_i = 0 \ldots .(1)$$

Take any point $Q(x_i)$ on the base. Then the co-ordinates of any point $P(x_i')$ on the line A_0Q may be given by

$$x_0 = \lambda x'_0 + \mu, \; x_1 = \lambda x_1', \; x_2 = \lambda x_2', \; x_3 = \lambda x_3', \; x_4 = \lambda x_4' \ldots .(2)$$

Since Q is a point on the quadric (1), its co-ordinates must satisfy the equations (1). Hence, eliminating λ and μ between equations (1) and (2), we obtain the locus of the point P (x_i') defined by the equation

$$\alpha_0^2 \sum_0^4 \sum_0^4 a_{ij} x_i x_j - 2\alpha_0 \sum_0^4 a_i x_i \sum_0^4 a_{0i} x_i + a_{00} \left(\sum_0^4 \alpha_i x_i \right)^2 = 0. \qquad (156.1$$

It will be seen that this equation is homogenous in x_1, x_2, x_3, x_4 and is free from x_0, so that it reduces to the form

$$\alpha_0^2 \sum_1^4 \sum_1^4 a_{ij} x_i x_j - 2\alpha_0 \sum_1^4 a_i x_i \sum_1^4 a_{0i} x_i + a_{00} \left(\sum_1^4 \alpha_i x_i \right)^2 = 0. \qquad (156.2$$

This locus is called a *Hypercone of the first species*, as distinct from a Hypercone of the second species, which will be described presently.

A hypercone of the first species may also be generated in another manner:

Let two fixed planes intersecting in a point O be defined by their generating hyperplanes (§ 18)

$$A + \lambda B = 0, \quad \text{and} \quad C + \mu D = 0. \qquad \ldots(3)$$

If these generating hyperplanes are made to correspond homographically, the locus of the planes of intersection of the corresponding hyperplanes is a hypercone of the first species, having the point O for vertex. This hypercone, in fact, consists of two systems of planes, one system has no other point common than the point O, and the other system mutually intersect in lines.

Since the generating hyperplanes are homographically related, λ and μ must be connected by a relation of the form

$$\alpha\lambda\mu + \beta\lambda + \gamma\mu + \delta = 0, \qquad \ldots(4)$$

whence, eliminating λ, μ, the required locus is obtained in the form—

$$\alpha.(-\frac{A}{B})(-\frac{C}{D}) + \beta(-\frac{A}{B}) + \gamma(-\frac{C}{D}) + \delta = 0,$$

$$i.e., \quad \alpha.AC - \beta.AD - \gamma.BC + \delta.BD = 0 \qquad (156.3$$

which is homogeneous in A, B, C, D and represents a hypercone, having the point O for vertex.

157. Hypercone of the Second Species, or a Double Cone :

Take a fixed line and a fixed conic, not in the same hyperplane. The locus of the planes drawn through the fixed line and the points of the conic is a *Hypercone of the second species* or *a Double Cone*, having the fixed line as the *vertex-edge* and the conic as the *directing curve*.

Again, if the base be itself a quadric cone V with the vertex O, and O′ be any other point lying outside of the hyperplane of the cone, then the hypercone generated by taking O′ as the vertex and V as the base is a hypercone of the second species.

From this definition it will be seen that this hypercone may be regarded in two different ways—the vertex O of the base in one case being the vertex of the hypercone in the other case. These two cones with vertices O and O′ evidently have a common base—a closed conic, and it follows then that a hypercone of the second species is determined by a closed plane curve and two points not lying in the same hyperplane with the conic, and is therefore called a *double cone*. The line OO′ joining the vertices is called the *vertex-edge* and the two cones are the *end-cones*.

The triangle determined by the two vertices and any point on the conic may be called the *element* of the hypercone.

A hyperplane section of the hypercone, not passing through both the vertices O,O′, is a cone, and the hypercone can be generated by taking this cone of section as base and any point on the line OO′ as vertex. In fact, if U be a plane section of the directing base cone V, not passing through the vertex, the hypercone can be generated by planes drawn through the vertex-edge OO′ and the points on U. Hence follows the earlier definition of a hypercone of the second species. *

* As an analytical locus, this may also be called a *plano-conical hypersurface*.

158. Condition for a Hypercone:

Let O (α_i) be the vertex and x'_i any point on the Hypercone defined by the general equation of the second degree

$$(a \between x)^2 = 0. \qquad \ldots(1)$$

Then, for all values of λ and μ, the point whose co-odinates are $\lambda x'_i + \mu\alpha'_i$ must lie on the Quadric (1).

Substituting these values for the co-ordinates in (1), we get

$$\lambda^2 (a \between x')^2 + 2\lambda\mu (a \between \alpha \between x') + \mu^2 (a \between \alpha)^2 = 0. \quad \ldots(2)$$

Since α_i and x'_i are two points on the quadric cone, we have

$$(a \between \alpha)^2 = 0 \quad \text{and} \quad (a \between x')^2 = 0$$

$\therefore \quad (a \between \alpha \between x') = 0$ for all positions of the point x'_i,

i.e., $a_{00}\,\alpha_0\,x'_0 + a_{11}\,\alpha_1\,x'_1 + \ldots\ldots\ldots\ldots$

$$+ a_{01}\,(\alpha_0\,x'_1 + \alpha_1\,x'_0) + \ldots\ldots = 0. \ldots(3)$$

for all values of x'_i $(i = 0, 1, 2, 3, 4)$.

Hence, the co-efficients in (3) must vanish identically. Eliminating the parameters $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ between these, the condition for a hypercone is obtained in the form of a determinant equation

$$\Theta \equiv \begin{vmatrix} a_{00} & a_{01} & a_{02} & a_{03} & a_{04} \\ a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \\ a_{20} & a_{21} & a_{22} & a_{23} & a_{24} \\ a_{30} & a_{31} & a_{32} & a_{33} & a_{34} \\ a_{40} & a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = 0. \qquad (158.1$$

Hence, the condition that the general equation of the second degree should represent a hypercone is that
$$\Theta=0.$$

As a particular case, the condition that the quadric $\Sigma\alpha_i x_i^2=0$ should represent a hypercone is obtained in the form

$$\Theta\equiv\alpha_0\alpha_1\alpha_2\alpha_3\alpha_4=0 \qquad (158.2$$

which requires that one or more of the co-efficients must vanish. If one co-efficient (say) $\alpha_0=0$, the polar hyperplane of any point x'_i is

$$\alpha_1 x_1 x'_1+\alpha_2 x_2 x'_2+\alpha_3 x_3 x'_3+\alpha_4 x_4 x'_4=0,$$

which evidently passes through the vertex $(1,0,0,0,0)$, but the polar hyperplane of this latter point is indeterminate. It can be easily seen that all lines drawn through A_0 to meet the quadric lie on the quadric, but all other lines meet it in two points at A_0, which is therefore a *double point* on the quadric. In this case, then, the quadric becomes a *hypercone of the first species* with A_0 as vertex, and its equation can be written as

$$\alpha_1 x_1{}^2+\alpha_2 x_2{}^2+\alpha_3 x_3{}^2+\alpha_4 x_4{}^2=0. \qquad (158.3$$

If two of the co-efficients, namely, $\alpha_0=\alpha_1=0$, the polar hyperplanes of all points pass through both the points A_0, A_1, *i.e.*, they all contain the line A_0A_1, whereas the polar hyperplane of any point on this line is indeterminate. In fact, A_0A_1 is a *double line* of the quadric, which can now be generated by means of planes drawn through A_0A_1. In this case, the quadric becomes a *hypercone of the second species* with A_0A_1 as vertex-edge, and its equation can be written as

$$\alpha_2 x_2{}^2+\alpha_3 x_3{}^2+\alpha_4 x_4{}^2=0. \qquad (158.4$$

159. Intersection of two Quadrics :

Let $\vartheta = (a \between x)^2 = 0$ and $\vartheta' = (a' \between x)^2 = 0$

represent two quadrics.

Then $\vartheta + \lambda\vartheta' = 0$ represents a Quadric Variety through the common points of ϑ and ϑ'.

This will represent a hypercone, if the condition (158.1) is satisfied. The necessary condition will be obtained by putting $a_{ij} + \lambda a'_{ij}$ for a_{ij} in the determinant equation $\Theta = 0$. The equation thus obtained will be of the fifth order in λ, and will therefore give as many values, for each of which $\vartheta + \lambda\vartheta' = 0$ will represent a hypercone.

Thus, in general, *five hypercones can be drawn through the common points of any two quadric varieties.*

It can easily be shown by taking the canonical form of the equation, that *the vertices of these five hypercones form a self-polar simplex with regard to all quadrics of the system.*

This is really an extension of the corresponding theorem in the ordinary space.

Let A_0 and A_1 be the vertices of two such hypercones, passing through the common points of two quadrics defined by

$$(a \between x)^2 = 0 \quad \text{and} \quad (a' \between x)^2 = 0. \qquad \ldots(1)$$

Then, if A_0 is the vertex of a self-polar simplex, the polar hyperplane of the vertex A_0 (x') with regard to both the quadrics must be identical, *i. e.*, the equations

$$(a \between x' \between x) = 0 \quad \text{and} \quad (a' \between x' \between x) = 0,$$

where x' corresponds to the co-ordinates of A_0, must be identical. That is

$$\left.\begin{aligned} a_{00}x_0 + a_{01}x_1 + a_{02}x_2 + a_{03}x_3 + a_{04}x_4 = 0 \\ a'_{00}x_0 + a'_{01}x_1 + a'_{02}x_2 + a'_{03}x_3 + a'_{04}x_4 = 0 \end{aligned}\right\} \quad \dots(2)$$

must be identical.

Similarly, since the polar hyperplanes of A_1 are also to be identical, we must have

$$\left.\begin{aligned} a_{10}x_0 + a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = 0 \\ a'_{10}x_0 + a'_{11}x_1 + a'_{12}x_2 + a'_{13}x_3 + a'_{14}x_4 = 0 \end{aligned}\right\} \quad \dots(3)$$

also identical.

From (2) it follows that

$$\frac{a_{00}}{a'_{00}} = \frac{a_{01}}{a'_{01}} = \frac{a_{02}}{a'_{02}} = \frac{a_{03}}{a'_{03}} = \frac{a_{04}}{a'_{04}}. \qquad \dots(4)$$

From (3), we have

$$\frac{a_{10}}{a'_{10}} = \frac{a_{11}}{a'_{11}} = \frac{a_{12}}{a'_{12}} = \frac{a_{13}}{a'_{13}} = \frac{a_{14}}{a'_{14}}. \qquad \dots(5)$$

Hence, either $a_{01} = 0 = a'_{01}$; or else if P be any point $\xi_i = \lambda x'_i + \mu x_i''$ on the line A_0A_1, then the two polar hyperplanes of P, *i.e.*,

$$(a \between \xi \between x) = 0 \quad \text{and} \quad (a' \between \xi \between x) = 0$$

must be identical.

The second alternative is evidently a particular case; consequently, the other is the only possible conclusion, and this shows that either of the points A_0, A_1 lies on the polar of the other with respect to either quadric. The same conclusion holds for the other vertices as well.

Hence, *the five vertices form a simplex, self-polar for any two quadrics, and for all quadrics drawn through the intersection of these two quadrics.*

160. Quadric Varieties of Revolution :

We have already seen that in the fourfold, rotation can take place around a plane as an axis-plane, or around a line as an axis-line. Hence, a hyperplane can also rotate about any of its planes or about any of its lines.

A quadric V_2^2 lying in the hyperplane, then, by rotating about one of its planes will generate a variety V_3^2 of revolution. Every point of it rotates in a plane absolutely orthogonal to the axis-plane and describes a circle, whose centre is the foot of the perpendicular drawn from the point on the axis-plane. Any section of this variety by any absolutely orthogonal plane of the axis-plane is a circle.

Suppose the equation of the generating (quadric) variety, referred to two orthogonal axes in the axis-plane, namely x, y, and one other, axis z, is written in the form

$$z^2 = f(x, y).$$

Then the equation of the (quadric) variety V_3^2 of revolution is obtained in the form

$$z^2 + w^2 = f(x, y). \tag{160.1}$$

This is called *a quadric variety of revolution of the first species*.

Again, a hyperplane can rotate about any axis-line, and it has two degrees of freedom of motion. Consequently, every point of it will generate a sphere. Hence, a quadric variety V_1^2 (a conic) lying in the hyperplane will generate, by rotation about the axis-line, a *quadric variety V_3^2 of revolution of the second species*.

The equation of the generating conic being taken as $y^2=f(x)$, the equation of the variety of revolution will be of the form

$$y^2+z^2+w^2=f(x). \tag{160.2}$$

161. The Hypersphere or the Spheric : *

The simplest of the quadric varieties of revolution is the *hypersphere*, or simply a *spheric*, which is the locus of a point in the fourfold always at a constant distance from a fixed point. The fixed point is called the *centre*, and the constant distance is called the *radius* of the hypersphere.

If the centre is at the fixed point $C(\alpha, \beta, \gamma, \delta)$ and the radius is denoted by r, the distance of a variable point $P(x, y, z, w)$ from C is given by

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2+(w-\delta)^2=r^2. \tag{161.1}$$

This, then, is the equation of the locus of P, *i.e.*, the hypersphere whose centre is the point C and radius r.

Since the equation involves five independent constants, a hypersphere is determined uniquely by five non-conjoint points. This is an equation of the second degree in the variables, in which the co-efficients of the square-terms are all equal and there are no terms containing the products of the variables. Hence, the general equation of the second degree in four variables will represent a hypersphere, when the co-efficients of x^2, y^2, z^2, w^2 are equal and the co-efficients of the

* We shall often denote a hypersphere by the term *spheric*.

products of variables are all zero. The general equation of a hypersphere can, therefore, be written in the form :

$$\Sigma x^2 + 2\Sigma \alpha x + C = 0 \tag{161.2}$$

or,

$$\Sigma(x+\alpha)^2 = \Sigma\alpha^2 - C \tag{161.3}$$

showing that it represents a hypersphere with centre at the point $(-\alpha, -\beta, -\gamma, -\delta)$, and radius $= \sqrt{\Sigma\alpha^2 - C}$.

When the origin is taken at the centre, the equation (161.1) reduces to

$$x^2 + y^2 + z^2 + w^2 = r^2.$$

Using homogeneous co-ordinates, the equation of a hypersphere can be written as

$$\Sigma x^2 + 2\Sigma a x x_0 + c x_0{}^2 = 0. \tag{161.4}$$

If the system of axes be oblique, the equation of a hypersphere takes the form

$$\sum_1^4 (x_i - \alpha_i)^2 + 2\sum_1^4 (x_i - \alpha_i)(x_j - \alpha_j)\cos\phi_{ij} = r^2 \tag{161.5}$$

where α_i are the co-ordinates of the centre and ϕ_{ij} is the angle between the axes of x_i and x_j. This can again be written in the normal form

$$\sum_1^4 x_i^2 + 2\sum_1^4 x_i x_j \cos\phi_{ij} + 2\sum_1^4 A_i x_i + B = 0. \tag{161.6}$$

The equation (161.6) represents a hypersphere, the centre and the radius may be determined by identifying it with (161.5), *i.e.*, we have then

$$-A_i = \sum_1^4 \alpha_j \cos\phi_{ij} \qquad (i = j \mp 1, 2, 3, 4)$$

$$B = \sum_1^4 \alpha_i^2 + 2\sum_1^4 \alpha_i\alpha_j \cos\phi_{ij} - r^2.$$

The first four equations give the co-ordinates of the centre, while the radius is determined by the last one.

The equation involves five parameters, and consequently, a hypersphere is determined by any five given

independent points. The equation may be obtained by eliminating these parameters from the equations (161.6) and five others obtained by substituting the co-ordinates of the five points in the same.

162. Two or more Spherics:

If $P(x_i)$ be any point in the fourfold, the left-hand side of the equation $(161.5) = PO^2 - r^2$, where O is the centre of the spheric. If Q and Q′ are the two points in which any secant drawn through P meets the spheric, then PQ. PQ′ is found to be equal to $PO^2 - r^2$.

This expression $PQ.PQ' = PO^2 - r^2$ is called the *Power* of the point P with respect to the spheric.

If, then, $S_1 = 0$ and $S_2 = 0$ are the equations of two spherics in the normal form, then $S_1 - S_2 = 0$ represents the locus of a point P, which has the same power with respect to each of the spherics, and consequently, with respect to each spheric of the pencil

$$\lambda_1 S_1 + \lambda_2 S_2 = 0.$$

Since the second degree terms on both sides cancel, the locus of P is represented by a linear equation and is, therefore, a hyperplane, which also contains the common points of the two spherics, if any.

The angle ψ at which the two spherics, namely,

$$\sum_1^4 x_i^2 + 2\sum_1^4 a_i x_i + \alpha = 0 \quad \text{and} \quad \sum_1^4 x_i^2 + 2\sum_1^4 b_i x_i + \beta = 0$$

intersect can be easily calculated and is given by the formula

$$\operatorname{Cos} \psi = \frac{\alpha + \beta - 2\Sigma\, a_i b_i}{(\sqrt{\Sigma a_i^2 - \alpha})\,(\sqrt{\Sigma\, b_i^2 - \beta})}. \qquad (162.1$$

They intersect at right angles, when $\psi = \pi/2$, and the condition for this is—

$$\alpha + \beta = 2\Sigma a_i b_i. \tag{162.2}$$

The condition that the two spherics will touch each other is obtained, by putting $\psi = \pi$, in the form—

$$(\alpha + \beta - 2\Sigma\, a_i b_i)^2 = 4(\Sigma\, a_i^2 - \alpha)(\Sigma\, b_i^2 - \beta). \tag{162.3}$$

163. The Sphere at Infinity :

Any hyperplane intersects the hypersphere in a sphere. If we consider the section by the hyperplane $w = k$, it is found that it meets the hypersphere in a sphere of radius $\sqrt{r^2 - k^2}$, which gradually diminishes in length as k increases, *i.e.*, as the hyperplane is moved parallel to itself, until when $k = r$, the section reduces to a point-sphere. If the hyperplane moves further away, the section becomes an imaginary sphere, until when it moves off to infinity, *i.e.*, when $k = \infty$, the section becomes an *imaginary sphere at infinity*. This sphere is the analogue of the *circular points at infinity* in a plane, and the *circle at infinity* in a hyperplane, and is called the *sphere at infinity* in the fourfold.

The equation (161.4) reduces to

$$x_0 = 0, \quad x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0.$$

It may be remarked that any plane has a line at infinity, which meets this sphere in two points, *i.e.*, the circular points at infinity in that plane; any hyperplane has a plane at infinity, which meets this sphere in the circle at infinity in that hyperplane. This sphere at infinity together with the hyperplane at infinity taken

twice, constitutes a degenerate quadric variety as an envelope and a locus respectively and is called the *Absolute*.*

164. The circumscribing Hypersphere of a Simplex:

The radius of the hypersphere circumscribing the simplex (pentahedroid) of five points can be easily determined from the identical relation satisfied by the mutual distances between six points in the fourfold (§ 30).

The centre of the circumscribing hypersphere and the five vertices of the pentahedroid must satisfy the relation obtained in § 30, and the co-ordinates of the centre are linear functions of those of the vertices.

Let the five vertices and the centre be denoted respectively by the numerals 1, 2, 3, 4, 5, 6, the sixth point being the centre.

Following the notation $(1,2)^2=(2,1)^2\equiv$square of the distance between the points denoted by the numbers 1 and 2, and denoting the radius by R, we have

$$R^2=(1,6)^2=(2,6)^2=(3,6)^2=(4,6)^2=(5,6)^2.$$

If, now, we put

$$(1,2)^2=c^2,\quad (1,3)^2=b^2,\quad (2,3)^2=a^2,\quad (1,4)^2=f^2,$$

$$(2,4)^2=g^2,\quad (3,4)^2=h^2,\quad (1,5)^2=l^2,\quad (2,5)^2=m^2,$$

$$(3,5)^2=n^2 \quad\text{and}\quad (4,5)^2=p^2$$

* The notion of sphere at infinity has often been used by the Italian School and denoted by J_2. Giacomini, *Sulla correspondenza fra la Geometria conforme di S_4 et la geometria projettiva dello spazio ordinario*,—Annali della Sc. Norm. Sup. di Pisa (1899).

the relation (30.1) can be written in the form

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 & f^2 & l^2 & R^2 \\ 1 & c^2 & 0 & a^2 & g^2 & m^2 & R^2 \\ 1 & b^2 & a^2 & 0 & h^2 & n^2 & R^2 \\ 1 & f^2 & g^2 & h^2 & 0 & p^2 & R^2 \\ 1 & l^2 & m^2 & n^2 & p^2 & 0 & R^2 \\ 1 & R^2 & R^2 & R^2 & R^2 & R^2 & 0 \end{vmatrix} = 0. \qquad (164.1$$

This, when simplified, can be written as

$$2R^2 \begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 & f^2 & l^2 \\ 1 & c^2 & 0 & a^2 & g^2 & m^2 \\ 1 & b^2 & a^2 & 0 & h^2 & n^2 \\ 1 & f^2 & g^2 & h^2 & 0 & p^2 \\ 1 & l^2 & m^2 & n^2 & p^2 & 0 \end{vmatrix} + \begin{vmatrix} 0 & c^2 & b^2 & f^2 & l^2 \\ c^2 & 0 & a^2 & g^2 & m^2 \\ b^2 & a^2 & 0 & h^2 & n^2 \\ f^2 & g^2 & h^2 & 0 & p^2 \\ l^2 & m^2 & n^2 & p^2 & 0 \end{vmatrix} = 0 \qquad (164.2^*$$

whence the required radius can be calculated as the ratio of two determinants.

Note. The method adopted is perfectly general and can be extended to spaces of any number of dimensions. Expressions for the radius of the circumscribing circle of a triangle and the circumscribing sphere of a tetrahedron are obtained as particular cases.

* Compare Frost, *Solid Geometry*, § 589.

165. The Tangent Hyperplane:

Any plane section of a hypersphere is a circle, which is the largest when the plane passes through the centre of the hypersphere. Any hyperplane section is a sphere, which is the largest when the hyperplane passes through the centre. The tangent line to the plane-section at any point is called a *tangent line* to the hypersphere. Similarly, any tangent plane to the hyperplane-section is a *tangent plane* to the hypersphere.

The tangent lines to a hypersphere at any point are all perpendicular to the radius drawn to the point and consequently lie in one and the same hyperplane, which is called the *tangent hyperplane*. It may be noted that the hypersphere lies on one side of this tangent hyperplane. All the planes situated in this tangent hyperplane and passing through the point of contact touch the hypersphere at that point and are called *tangent planes* to the hypersphere.

A tangent hyperplane is, therefore, defined as the locus of all lines at right angles to the radius drawn through the point.

Let the hypersphere be defined by the equation

$$x^2+y^2+z^2+w^2=r^2. \qquad \ldots(1)$$

The equations of the radius drawn to any point $P(x',y',z',w')$ on the hypersphere are

$$\frac{x}{x'}=\frac{y}{y'}=\frac{z}{z'}=\frac{w}{w'}. \qquad \ldots(2)$$

The equations to any line through $P(x',y',z',w')$ are

$$\frac{x-x'}{l}=\frac{y-y'}{m}=\frac{z-z'}{n}=\frac{w-w'}{p}, \qquad \ldots(3)$$

where l, m, n, p are the assumed direction-cosines of the line. By the condition of orthogonality of the lines (2) and (3) we have

$$lx' + my' + nz' + pw' = 0.$$

or, $$x'(x-x') + y'(y-y') + z'(z-z') + w'(w-w') = 0.$$

Since (x', y', z', w') lies on the hypersphere, this reduces

to $$xx' + yy' + zz' + ww' = x'^2 + y'^2 + z'^2 + w'^2 = r^2. \quad (165.1$$

This is the locus of the line (3) and is a hyperplane, which is the *tangent hyperplane* at the point P (x', y', z', w'). This method can be applied in any case when the hypersphere is defined by other forms of equations.

The condition that any given hyperplane

$$lx + my + nz + pw = k$$

may touch the hypersphere may be obtained in the usual way.

166. Hyperplane-Sections of a Hypersphere:

The canonical form of the equation of a hypersphere, when the origin is the centre, may be assumed for purposes of analytical investigation of its sections by planes or hyperplanes in general position. We accordingly write the equation of the hypersphere in the form

$$x^2 + y^2 + z^2 + w^2 = r^2. \quad \ldots \quad (1)$$

The hyperplane $w = k$ intersects it in the sphere

$$x^2 + y^2 + z^2 = r^2 - k^2, \quad w = k. \quad \ldots \quad (2)$$

This shows that all sections of the hypersphere by hyperplanes parallel to $w=0$ are spheres and this is true of any hyperplane section of the hypersphere.

Let O be the centre and Q its projection on the hyperplane of section. If, then, P is any point on the sphere of section, then OP = radius r, and PQ = the radius of the sphere of section.

Hence, $OP^2 = PQ^2 + OQ^2$,

or, $$PQ^2 = OP^2 - OQ^2 = \text{ constant}, \qquad \dots \quad (3)$$

i.e., the length PQ is constant for all variable positions of P on the sphere.

$\therefore$ The point Q, which is the projection of O, the centre of the hypersphere, on the hyperplane of section, is the centre of the sphere of section. Hence, we conclude that a *hyperplane-section of a hypersphere is a sphere, whose centre is the projection of the centre of the hypersphere on the hyperplane of the sphere.*

When, however, the hyperplane passes through the centre, the section is called a *great sphere*, its centre coinciding with that of the hypersphere.

Cor. 1: Any four non-coplanar points on the hypersphere determine a hyperplane whose section on the hypersphere is a sphere. Any three points on the hypersphere and its centre will determine a great sphere.

Cor. 2: Two spheres on the same hypersphere intersect in a circle in the plane of cleavage of the two hyperplanes of the spheres.

167. Plane Sections of a Hypersphere :

The section of the hypersphere (1) by the plane $z=0$, $w=0$ is given by $x^2+y^2=r^2$, $z=0$, $w=0$, which is evidently a circle in the plane (x,y).

This is true of all plane-sections, and since only one perpendicular can be drawn from the centre of the hypersphere on any plane, if Q be the foot of this perpendicular and P any point on the circle of section, we have, as above,

$$PQ^2=OP^2-OQ^2=\text{constant},$$

showing that Q is the centre of the circle of section.

Thus, *any plane-section of a hypersphere is a circle, whose centre is the projection of the centre of the hypersphere on the plane of section.*

When, however, the plane passes through the centre of the hypersphere, $OQ=0$, and the section is a *great circle* on the hypersphere.

It must be observed that a plane in general position will not intersect the hypersphere, if it has not more than one point on the hypersphere.

Cor. : Any three points on the hypersphere determine a plane which intersects the hypersphere in a circle. Any two points on the hypersphere and its centre determine a plane, which cuts the hypersphere in a great circle.

168. Circles on the Hypersphere :

Consider two planes of general position, passing through the centre of the hypersphere. These two planes have no other common point beside the centre of the hypersphere. Hence, their circles of section on th hypersphere have no common point at all.

Hence, *any two great circles on a hypersphere do not intersect*. If, however, the two planes through the centre lie in one and the same hyperplane, they have a diameter common between them, and their great circles of section lie on the same great sphere and intersect in the extremities of the common diameter.

Conversely, if two great circles intersect, their planes lie in one and the same hyperplane through the centre, which intersects the hypersphere in a great sphere on which the two great circles lie.

Again, consider a plane and a hyperplane drawn through the centre of the hypersphere. They intersect in a diameter of the hypersphere. Consequently, the great circle and the great sphere intersect in the extremities of this common diameter only.

Hence, *a great circle and a great sphere on the hypersphere intersect in only two points, namely, the extremities of a diameter.*

Note. Two great circles on the same great sphere intersect in the extremities of a diameter, but two great circles on a hypersphere do not generally intersect.

169. Polar Properties:

Consider a pencil of hyperplanes $A+\lambda B=0$, having the common vertex-plane (plane of cleavage) α exterior to the hypersphere.

It is easily seen that only two hyperplanes of the pencil touch the hypersphere at two points P and Q (say). Any hyperplane through P, Q and the centre O will intersect the hypersphere in a great sphere $\mathfrak{S}$. It will intersect the plane α in a line L (say) and the two tangent

hyperplanes in two planes touching the great sphere at the two points P and Q. We may say that the lines L and PQ are *conjugate* with respect to the great sphere $\mathfrak{S}$, since L is the line of intersection of the tangent planes at P and Q, and PQ is perpendicular to the plane determined by L and the centre, and they are also mutually perpendicular.

Hence follows that all the lines lying in α are conjugate to the line PQ. The plane α and the line PQ may be said to be *conjugate* with respect to the Hypersphere.

The tangent hyperplane at any point A (x', y', z', w')

on the hypersphere $\qquad x^2+y^2+z^2+w^2=r^2$

is given by $\qquad xx'+yy'+zz'+ww'=r^2. \qquad \ldots(1)$

If this passes through a fixed point P $(\alpha, \beta, \gamma, \delta)$

we have $\qquad \alpha x'+\beta y'+\gamma z'+\delta w'=r^2. \qquad \ldots(2)$

If, now, P $(\alpha, \beta, \gamma, \delta)$ remains fixed and (x', y', z', w') varies, the locus of A (x', y', z', w'), *i.e.*, the point of contact is given by

$$\alpha x+\beta y+\gamma z+\delta w=r^2 \qquad \ldots(3)$$

which represents a hyperplane, and this is called the *polar hyperplane* of the point P$(\alpha, \beta, \gamma, \delta)$ with respect to the hypersphere. The point and the polar hyperplane may be said to be *conjugate*. It is evident that the line joining the point P to the centre of the hypersphere is normal to the polar hyperplane (3).

If, again, the tangent hyperplane passes through another fixed point Q $(\alpha', \beta', \gamma', \delta')$, we have

$$\alpha' x'+\beta' y'+\gamma' z'+\delta' w'=r^2 \qquad \ldots(4)$$

and it follows then that the hyperplane (1) passes through all the points lying on the line PQ.

For, from (2) and (4), we get

$$(\alpha+\lambda\alpha')x'+(\beta+\lambda\beta')y'+(\gamma+\lambda\gamma')z'+(\delta+\lambda\delta')w'=r^2+\lambda r^2. \quad \ldots(5)$$

Therefore, the locus of (x', y', z', w') is—

$$(\alpha+\lambda\alpha')x+(\beta+\lambda\beta')y+(\gamma+\lambda\gamma')z+(\delta+\lambda\delta')w=r^2+\lambda r^2 \quad \ldots(6)$$

for all values of λ.

Hence, the locus of A is given by

$$\alpha x+\beta y+\gamma z+\delta w=r^2, \qquad \alpha' x+\beta' y+\gamma' z+\delta' w=r^2 \quad \ldots(7)$$

which represent the two polar hyperplanes of P and Q. These intersect in a plane which meets the hypersphere in a circle. Hence, *the singly infinite system of tangent hyperplanes drawn through any straight line touch the hypersphere at points of a circle. The plane of this circle is conjugate to the line.*

The line and the plane may be said to be *conjugate* with respect to the hypersphere.

Thus, points, lines, planes and hyperplanes have got respectively hyperplanes, planes, lines and points for their conjugate elements with respect to a hypersphere, or more generally *w. r. t.* any quadric variety.

170. Right Hypercones of the first Species:

If the directing surface be a sphere, and the points of this sphere be joined to a point O, not lying in the hyperplane of the sphere, the joining lines will generate a hypercone of the first species.

Or, if through any point external to a hypersphere, the ∞^2 system of tangent lines be drawn, they will generate a hypercone of the first species.

In the first case, the hypersurface is called a *spherical Hypercone*, having the line joining the vertex with the centre of the base as *axis*. If, however, the axis is at right angles to the hyperplane of the base, the hypercone is called a *right circular*, or simply, a *right hypercone* of the first species.

In the second case, the hypercone is called a *right circular*, or simply, a *right hypercone*, having the line joining the vertex to the centre of the hypersphere as the *axis-line*. The points of contact of the tangent lines lie on a spherical surface at right angles to the axis and is called the *sphere of contact*. In fact, the vertex is the *pole* and the base is the *polar* hyperplane with respect to the hypersphere.

171. Generation of a right Hypercone of the first Species:

A right hypercone of the first species can be generated by the motion of a right-angled triangle in the fourfold.

Let ABC be a triangle right-angled at C. Keeping BC fixed, if the triangle ABC is allowed to move in all possible manner in the fourfold, the side CA will generate a hyperplane orthogonal to BC at C, and since AC is constant, A is always at the same distance from C, *i.e.*, the locus of A is a sphere with centre C. Hence, the hypotenuse BA will generate a right hypercone of the first species with CB as axis, B as *vertex* and the locus of C as the *base* or the guiding sphere.

Let C be taken as the origin and CB as the axis of w. Also let $CB=a$ and $CA=r$. Then, the co-ordinates of B are $(0,0,0,a)$, and suppose the co-ordinates of A are $(\alpha, \beta, \gamma, 0)$. If P (x, y, z, w) is any point on the side AB, we have

$$x=\mu\alpha, \qquad y=\mu\beta, \qquad z=\mu\gamma, \qquad w=\lambda a,$$

subject to the condition $\lambda+\mu=1$. ... (1)

We have $$\alpha^2+\beta^2+\gamma^2=AC^2=r^2. \qquad \text{... (2)}$$

Eliminating α, β, γ between the equations (1) and (2), we get

$$x^2+y^2+z^2=\mu^2 r^2 \quad \text{and} \quad a\mu=a-w \qquad \text{... (3)}$$

whence, eliminating μ, we obtain the locus of the point P (x, y, z, w) defined by the equation

$$a^2(x^2+y^2+z^2)=r^2(a-w)^2 \qquad (171.1$$

which represents a right hypercone of the first species, and is evidently a quadric variety of revolution.

172. Right Hypercone of the second Species:

If through a line exterior to a hypersphere, a singly infinite system of tangent planes be drawn, the points of contact of these tangent planes lie on the plane conjugate to the line with respect to the hypersphere. These tangent planes will generate a right hypercone of the second species.*

* Prperties of Hypercones of the second species have baen exhaustively studied by Veronese in his great work—Grundzüge, etc. § 149, pp. 557-60.

The plane passing through the given line and the centre of the hypersphere may be called a *plane of symmetry*. For, each generating hyperplane of this plane meets the generating (tangent) planes in lines, which are symmetrical, two by two, with respect to this plane. This plane of symmetry may be called the *axis-plane* and the line the *vertex-edge* of the hypercone. It follows further that each of the genarating planes is inclined to the axis-plane at the same angle. There is evidently another plane, the plane absolutely orthogonal to this plane of symmetry, to which the generating planes are also equally inclined.

From the foregoing properties it may be inferred that a plane, isocline to a given plane at a given point of it, will generate a right hypercone of the second species, of which the given plane is the axis-plane. This is analogous to the generation of an ordinary cone by the rotation of a line around a fixed line at one of its points, always making the same angle with it. Here, if at any point of a plane another plane be drawn isocline to the first, and if this latter plane is rotated around the given plane, remaining always inclined to it at the same angle, then a right hypercone of the second species will be generated. All these planes, however, will meet the given plane in the vertex-edge of the hypercone.

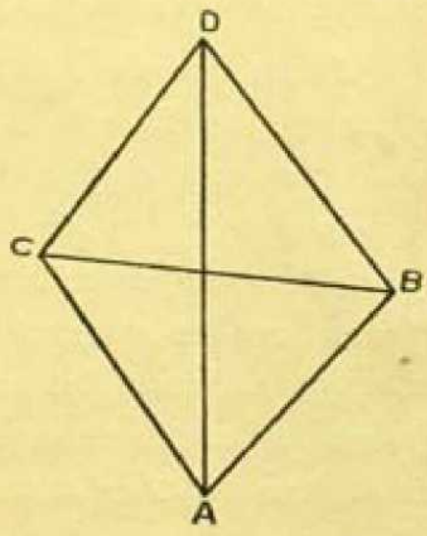
A right hypercone of the second species may be obtained in a different manner:

If a circle is taken as the base, then the planes drawn through the vertex-edge and the different points of the circle generate a circular hypercone of the second species, or a *circular double cone*. The triangle formed by the vertex-edge and the centre of the circle may

be called the *axis-element*. If the plane of the element is absolutely orthogonal to the plane of the base, the double cone is called a *right circular*, or simply, a *right double cone*. If the two vertices are at the same distance from the plane of the base, the double cone is said to be *isosceles*.

173. Generation of a right Hypercone of the second Species:

Let ABCD be a tetrahedron, the edge CD being at right angles to the face ABC at C. If now the face ABC be kept fixed and the tetrahedron be allowed to move in all possible manner, *i.e.*, if the tetrahedron be rotated around the plane ABC as the axis-plane, CD will rotate in the absolutely orthogonal plane to ABC at C, and D will then trace out in this plane a circle, having C for its centre. The plane DAB will always pass through the edge AB and through the point D of this circle. Hence, DAB may be taken as the generating plane of the hypercone, having AB as the vertex-edge, (A and B being the vertices of two other cones—the *end-cones*), and the locus of D as the base-circle. The lines AD and BD will generate the end-cones. The plane ABC is a plane of symmetry and the axis-element of the hypercone.



Let C be taken as origin and the plane ABC as the co-ordinate-plane of (z, w). Then D will move in the co-ordinate-plane of (x, y).

Let $DC = r$ and the co-ordinates of the points be denoted as

$$A(0, 0, \gamma, \delta), \quad B(0, 0, \gamma', \delta') \quad \text{and} \quad D(\alpha, \beta, 0, 0)$$

then, $$\alpha^2 + \beta^2 = r^2. \qquad \ldots \quad (1)$$

If, then, P (x, y, z, w) is any point in the plane ABD, we may take

$$\left.\begin{array}{ll} x = \nu\alpha & z = \lambda\gamma + \mu\gamma' \\ y = \nu\beta & w = \lambda\delta + \mu\delta' \\ \multicolumn{2}{l}{\text{subject to the condition } \lambda + \mu + \nu = 1 \ (\S 32).} \end{array}\right\} \quad \ldots \quad (2)$$

Now the locus of P may be obtained by eliminating $\alpha, \beta, \lambda, \mu, \nu$ between equations (1) and (2).

From (1) and (2), we get $x^2 + y^2 = \nu^2 r^2$. $\quad \ldots \quad (3)$

Also from (2), solving for λ and μ, we get

$$\lambda = \frac{z\delta' - w\gamma'}{\gamma\delta' - \gamma'\delta}, \quad \mu = \frac{w\gamma - z\delta}{\gamma\delta' - \gamma'\delta}. \qquad \ldots \quad (4)$$

Substituting these values, we get

$$\nu = 1 - \lambda - \mu = 1 - \frac{z\delta' - w\gamma'}{\gamma\delta' - \gamma'\delta} - \frac{w\gamma - z\delta}{\gamma\delta' - \gamma'\delta}$$

$$= \frac{(\gamma\delta' - \gamma'\delta) + (w\gamma' - z\delta') + (z\delta - w\gamma)}{\gamma\delta' - \gamma'\delta}$$

whence, $$x^2 + y^2 = r^2 \left\{ \frac{(\gamma\delta' - \gamma'\delta) + (w\gamma' - z\delta') + (z\delta - w\gamma)}{\gamma\delta' - \gamma'\delta} \right\}^2$$

or, $$(\gamma\delta' - \gamma'\delta)^2 (x^2 + y^2) = r^2 \{\gamma\delta' - \gamma'\delta) + (w\gamma' - z\delta') + (z\delta - w\gamma)\}^2$$

$$= r^2 \{z(\delta - \delta') + w(\gamma' - \gamma) + (\gamma\delta' - \gamma'\delta)\}^2$$

(173.1

which represents a right hypercone of the second species.

If, however, we make use of oblique axes, that is, if CA and CB are taken as the axes of z and w respectively, and the axes of x and y are taken in the absolutely orthogonal plane, inclined at an angle ω, and if $AC=b$, $BC=a$, $DC=r$, the equation of the hypercone may be obtained in a simpler form. We have

$$\alpha^2+\beta^2+2\alpha\beta\cos\omega=r^2,\ \ \gamma=b,\ \ \delta=0;\ \ \gamma'=0,\ \ \delta'=a \quad \dots (5)$$

whence from (1), (2) and (5), we get

$$a^2b^2(x^2+y^2+2xy\cos\omega)=r^2(az+bw-ab)^2. \qquad (173.2$$

174. Pole and Polar Circles :

A hyperplane intersects a hypersphere in a sphere. The diameter of the hypersphere, perpendicular to the hyperplane, is called the *axis* of the sphere, and its two extremities are called the *poles* of the sphere.

It can be proved, exactly as in the ordinary space, that a pole of a sphere is equally distant from every point of the surface of the sphere. For, if P be the pole and Q be any point on the surface of the sphere, the plane determined by the axis OP and the point Q meets the hypersphere in a great circle, the arc PQ of this circle is found to be constant for all positions of the point Q.* In particular, when the sphere is a great sphere, the pole P is at a quadrant's distance from all points on the surface of the sphere. These facts may be stated in a different form :

* I. Todhunter, *Spherical Trigonometry*, § 6, p. 3.

The locus of points on a hypersphere at a quadrant's distance from either of the two extremities of a diameter is a great sphere, having the diameter as its axis and the extremities as its poles.

Again, any given plane intersects the hypersphere in a circle. The plane drawn through the centre of the hypersphere, absolutely orthogonal to this plane, intersects the hypersphere in a great circle, which may be called the *polar great circle* of the given circle, and its plane the *axis plane* of the given circle. It is to be noted, that the two circles have no common points, and the plane of the polar great circle meets the plane of the given circle at the centre of the latter, or what is the same thing, the plane drawn through the centre of a plane section of a hypersphere, absolutely orthogonal to the latter, meets the hypersphere in the polar great circle. Any sphere of the hypersphere that passes through the given circle will intersect the polar great circle in two points A and B (say). These two points are then two opposite *poles* of the circle on the sphere drawn, since the line AB is normal to the plane of the circle. Consequently, each of A and B is equidistant from all the points of the circle. Hence, we conclude that

Each point of the polar great circle of a circle on a hypersphere is equidistant from all the points of the given circle.

In particular, when the plane of the given circle passes through the centre of the hypersphere, *i.e.*, when it is itself a great circle, it is the polar great circle of its polar great circle, and in this case, the distance between any two points, one in each of the

two polar great circles is a quadrant. In fact two absolutely orthogonal planes through the centre of the hypersphere meet it in two great circles which are mutually polar.

Hence, *the locus of points on a hypersphere, at a quadrant's distance from the points on the circumference of a great circle, is another great circle, which is the polar great circle of the former.*

Again, a plane and a hyperplane will meet the hypersphere in a circle and a sphere respectively, which meet each other in two points only, *i.e.*, the points in which the line common between the plane and the hyperplane meets the hypersphere. If, however, the plane is drawn through the centre orthogonal to the hyperplane, the great circle passes through the poles of the sphere.

Hence, *the great circles drawn through the poles of a sphere are perpendicular to the sphere; and conversely, any great circle perpendicular to a sphere passes through the poles of the sphere.*

Thus, it appears that a reciprocal relation of pole and polar may be established between points and great spheres and between great circles on the hypersphere. Points may be made to correspond to their polar great spheres, and a great circle, regarded as locus of points, may be made to correspond to another great circle, regarded as common between the polar great spheres of the points on the first.

All the above facts stated geometrically may be analytically demonstrated by taking the canonical form of the equation of the hypersphere and the planes and hyperplanes drawn through its centre.

175. Hyperspherical Angles:

The nature of the geometry on the hypersphere, as discussed above, necessitates introduction of new concepts regarding angular magnitudes, which may be taken as measures of the inclinations of spherical elements on the hypersphere. Two such notions may be postulated corresponding to the spherical dihedral angle and the spherical trihedral angle of the ordinary space, namely, the *hyperspherical dihedral angle* and the *hyperspherical trihedral angle*. These are, however, mere abstractions, but all the same, their postulation will much facilitate the study of the geometry of the hypersphere, showing how these notions can be extended for wider applica tions.

(1) Hyperspherical Dihedral Angle:

Two hyperplanes drawn through the centre of a hypersphere will meet this latter in two great spheres with a common great circle, in which the plane of cleavage of the two hyperplanes meets the hypersphere. Thus, there are two great hemispheres, not belonging to the same great sphere, having a common great circle in the plane of cleavage of their hyperplanes. They are said to form what may be called a *hyperspherical dihedral angle* with the great circle as a common face. This is quite analogous to the spherical dihedral angle of the ordinary space.*

* Just as two great semi-circles enclose an area (a lune) on the surface of a sphere, similarly in the fourfold, two great hemispheres with a common edge enclose a definite volume, like a double convex lens, and the inclination of the two hemispheres is defined as a hyperspherical dihedral angle.

At any point of the common edge, the tangent half-planes to the two great spheres intersect in the tangent line of the edge and form a dihedral angle between them. This dihedral angle has a constant measure for all positions of the point on the common edge, and can, therefore, be taken as a measure of the hyperspherical dihedral angle. For, the two normals to the hyper-planes of the two great hemispheres have fixed directions, and consequently they include a constant plane angle, which also measures the dihedral angle between the two tangent planes. Hence, we conclude that a *hyperspherical dihedral angle has the same measure at all points of its edge* (compare §§ 62 and 65).

(2) HYPERSPHERICAL TRIHEDRAL ANGLE :

Three hyperplanes drawn through the centre of a hypersphere will meet it in three different great spheres meeting in the two extremities of the diameter common to the three hyperplanes. At each of these points, then, three great hemispheres mutually meet in three great circles (not belonging to the same sphere). These three great circles meeting in a common point are said to form what may be conceived as a *hyper-spherical trihedral angle*, having the three spherical angles on the three great hemispheres as *face-angles* and the three arcs as the *edges*. This is analogous to a solid angle in the ordinary space. The three great circles meeting in one point P also meet in the diametrically opposite point P′, enclosing a portion of the boundary-content of the hypersphere, which may be called a *hyperlune*, corresponding to a lune on a sphere in the ordinary space. The three tangent lines form a solid angle at the point, which may be taken as a

measure of the *hyperspherical trihedral angle*. Volumes of two hyperlunes having congruent trihedral (solid) angles at a vertex, may be compared, and it may be postulated that two hyperlunes having equal spherical trihedral (solid) angles are congruent, and that volumes of hyperlunes are proportional to their hyperspherical trihedal angles.

Having premised these, we may consider the whole boundary-content of a hypersphere as a hyperlune with a spherical trihedral angle, measured by the solid angle formed at any point, *i. e.*, equal to eight right solid angles (4π), and the volume of a hyperlune with a spherical trihedral angle, whose radian measure is $\mathbf{A}$ (solid), can be calculated thus:

$$\frac{\text{Volume of the Hyperlune}}{\text{Boundary-content of the Hypersphere}} = \frac{\mathbf{A}}{4\pi} \qquad (175.1$$

If, then, r is the radius of the hypersphere, the boundary-content of the hypersphere is $2\pi^2 r^3$ (see § 177). Then, the Volume of the Hyperlune

$$= \frac{\mathbf{A}}{4\pi} \cdot 2\pi^2 r^3 = \tfrac{1}{2}\mathbf{A}\pi r^3. \qquad (175.2$$

Note.—We shall often use the word *spherical* in place of *hyperspherical*, where no ambiguity arises.

176. Spherical Simplex :

Let A, B, C, D be four non-coplanar points on a hypersphere, but not lying on the same great sphere. If these points be joined, two by two, by arcs of great circles, the configuration obtained may be called a *hyperspherical tetrahedron*, or, a *spherical simplex*,* a conception formed in analogy to a spherical triangle on a sphere.

* P. H. Schoute, *loc. cit.*, Part II, § 97, p. 291.

Each group of three points determines a great sphere, so that a spherical simplex has four faces, each a spherical triangle on a great sphere. Hence, a spherical simplex consists of four points, not points of one great sphere, and all points between any two of these four points lying on that portion of the great circle which does not contain their opposite points, and all points of the four spherical triangles determined by each group of three points. Thus, a spherical simplex has *six edges*, each less than 180° along a spherical dihedral angle, *four faces*, which are four spherical triangles on four different great spheres, and *four vertices* with four spherical trihedral angles.

The determination of the content of the spherical simplex is one of the most difficult problems in the fourfold. Schläfli * has shown that the content of a spherical simplex $S_p(2n+1)$ can be expressed by means of the contents of the bounding spherical simplexes of lower orders, but there is no simple method of calculating the content of a spherical simplex $S_p(2n)$.† This can also be inferred from the fact that the extended form of Euler's theorem for a convex *polytope* ‡ in n-dimensional space is homogeneous, when n is even, and non-homogeneous, when n is odd. In the case of a spherical simplex in the fourfold, the content can be calculated for very special forms.

Even if *spherical trihedral* angles are conceived, the content of a spherical simplex cannot be obtained in a simple form analogous to that of a spherical triangle.

* L. Schläfli, *loc. cit.*, § 22, p. 67.

† A theorem by H. Poincaré, *Comptes Rendus*, Vol. 140, p. 113 (1905), may be consulted.

‡ See § 191.

Suppose A$'$, B$'$, C$'$, D$'$ are the opposite points of A, B, C, D respectively. There are then eight pairs of simplexes associated with ABCD, each pair symmetrically situated with respect to the centre and their volumes, therefore, congruent.* Denoting the four vertices and their opposites respectively by the numerals 1, 2, 3, 4 and 1$'$, 2$'$, 3$'$, 4$'$, the eight congruent pairs are—

1234, 1$'$2$'$3$'$4$'$; 1$'$234, 12$'$3$'$4$'$; 12$'$34, 1$'$23$'$4$'$; 123$'$4, 1$'$2$'$34$'$;
1234$'$, 1$'$2$'$3$'$4 ; 123$'$4$'$, 1$'$2$'$34 ; 1$'$23$'$4, 12$'$34$'$; 1$'$234$'$, 12$'$3$'$4.

If $V, V_1, V_2, V_3, V_4, V_{12}, V_{13}, V_{14}$ respectively denote their contents, it is seen that they together make up the boundary-content of the hyperhemisphere, and V forms a hyperlune with each of V_1, V_2, V_3, V_4.

$$\therefore \quad \pi^2 r^3 = V + V_1 + V_2 + V_3 + V_4 + V_{12} + V_{13} + V_{14}$$

$$\text{or,} \quad 3V + \pi^2 r^3 = (V + V_1) + (V + V_2) + (V + V_3) + (V + V_4) + \Sigma V_{12}$$

$$= \tfrac{1}{2}\pi r^3(\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D}) + \Sigma V_{12}$$

$$i.e., \quad V = \tfrac{1}{6}\pi r^3(\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D} - 2\pi) + \tfrac{1}{3}\Sigma V_{12},$$

where **A**, **B**, **C**, **D** are the spherical trihedral angles at A, B, C, D, measured in radians.

When, therefore, the volume of one of these tetrahedrons is given, the volumes of others can be determined.

177. The Boundary-content of a Hypersphere :

The boundary-content of a hypersphere can be calculated exactly in the same manner as the circumference of a circle, or the surface of a sphere, is found in the lower spaces.

* I. Todhunter, *loc. cit.*, § 33.

Defining the hypersphere by the equation

$$x^2+y^2+z^2+w^2=r^2 \tag{1}$$

we may consider the section made by a hyperplane at right angles to the axis of w at a distance w from the centre. The section is evidently a sphere of radius $\sqrt{r^2-w^2}$, and consequently its surface is $4\pi(r^2-w^2)$.

If, now, we consider a thin hyperspherical disc of small surface ds, having this sphere for base, the boundary-content of this thin disc $=4\pi\ (r^2-w^2)\ ds$.

If ξ be the radius of the sphere of section, we have

$$\xi^2=r^2-w^2, \quad \text{whence} \quad \frac{d\xi}{dw}=-\frac{w}{\xi}$$

and
$$ds^2=d\xi^2+dw^2=\left\{1+\left(\frac{d\xi}{dw}\right)^2\right\}dw^2$$

$$\therefore \quad ds=\frac{ds}{dw}.dw=\sqrt{1+\left(\frac{d\xi}{dw}\right)^2}\ .\ dw$$

$$=\frac{r}{\sqrt{r^2-w^2}}.\ dw$$

$\therefore$ The boundary-content S_4

$$=2\int_0^r 4\ \pi.\ (r^2-w^2).\frac{r}{\sqrt{r^2-w^2}}.\ dw$$

$$=8\ \pi \int_0^r r\sqrt{r^2-w^2}.\ dw$$

$$=8\ \pi\int_0^{\frac{\pi}{2}} r^3 \cos^2\theta\ d\theta, \quad \text{putting } w=r\sin\theta$$

$$=8\ \pi\ r^3.\ \tfrac{1}{2}.\ \frac{\pi}{2}=2\pi^2\ r^3. \tag{177.1}$$

Making use of a triple integral, the boundary-content may be obtained from

$$\iiint\sqrt{1+\left(\frac{dw}{dx}\right)^2+\left(\frac{dw}{dy}\right)^2+\left(\frac{dw}{dz}\right)^2}\ dx\ dy\ dz$$

If, then, we put

$$x = r\cos\alpha, \qquad y = r\sin\alpha\cos\beta$$
$$z = r\sin\alpha\cos\beta\cos\gamma, \qquad w = r\sin\alpha\sin\beta\sin\gamma$$

the above integral takes the form

$$\iiint r^3\sin^2\alpha\sin\beta\,d\alpha\,d\beta\,d\gamma$$

which, taken between proper limits, gives the result $2\pi^2 r^3$.

178. The Hypervolume of a Hypersphere:

Proceeding as above, the radius of the sphere made by a hyperplane at a distance w from the centre is $\sqrt{r^2-w^2}$, and its volume is, therefore, $\frac{4}{3}\pi(r^2-w^2)^{\frac{3}{2}}$.

Considering a thin hyperspherical disc of small thickness dw, having this sphere as base, its content

$$=\tfrac{4}{3}\pi(r^2-w^2)^{\frac{3}{2}}\,dw.$$

To obtain the hypervolume V_4 of the entire hypersphere, we integrate this between the limits $-r$ to $+r$, or, we may take twice the integral between the limits from 0 to r, *i.e.*,

$$\text{The content } V_4 = 2\int_0^r \tfrac{4}{3}\pi\,(r^2-w^2)^{\frac{3}{2}}\,dw$$

$$=\tfrac{8}{3}\pi r^4\int_0^{\frac{\pi}{2}}\cos^4\theta\,d\theta, \quad \text{putting } w = r\sin\theta$$

$$=\tfrac{8}{3}\pi r^4\cdot\tfrac{3}{4}\cdot\tfrac{1}{2}\cdot\tfrac{1}{2}\pi = \tfrac{1}{2}\pi^2 r^4. \qquad (178.1$$

Note.—The word *content* may conveniently be used to denote the *hypervolume* of a four-dimensional configuration.

The hypervolume may be calculated in another manner, from the fact that it is proportional to r^4, where r is the radius of the hypersphere.

$$\therefore \quad V_4 = k_4 . r^4 \qquad \dots \quad (1)$$

where k_4 is some constant to be determined.

Dividing the hypersphere into thin discs by a series of hyperplanes, since each section is a sphere, we have

$$V_4 = k_4 . r^4 = 2\int_0^r k_3 .(r^2 - w^2)^{\frac{3}{2}} . dw$$

$$\therefore \quad k_4 = 2k_3 \int_0^{\frac{\pi}{2}} \sin^4 \theta \, d\theta, \quad \text{putting } w = r \cos \theta$$

$$= 2k_3 . \tfrac{3}{4} . \tfrac{1}{2} . \tfrac{\pi}{2}, \quad i.e., \quad k_4 = k_3 . 2 . \tfrac{3}{4} . \tfrac{1}{2} . \tfrac{\pi}{2} \quad \dots \quad (2)$$

But, since $\quad k_3 = \frac{4}{3}\pi, \quad$ we have—

$$k_4 = 2 . \tfrac{3}{4} . \tfrac{1}{2} . \tfrac{\pi}{2} . \tfrac{4}{3}\pi \qquad \therefore \quad k_4 = \tfrac{1}{2}\pi^2$$

$$\therefore \quad V_4 = k_4 . r^4 = \tfrac{1}{2}\pi^2 r^4. \qquad (178.2$$

From what has been stated above, it now at once follows that the hypervolume or content

$$V_4 = \frac{1}{2}\pi^2 r^4 = \frac{r}{4} . 2\pi^2 r^3 = \frac{r}{4} . S_4,$$

i.e., the hypervolume of a hypersphere is equal to its boundary-content multiplied by one-fourth the radius of the hypersphere.

In fact, a similar relation can be noticed between the area V_2 and the circumference S_2 of a circle in a

plane, or, the volume V_3 and the surface S_3 of a sphere in the ordinary space. We have

$$V_2=\pi r^2, \quad S_2=2\pi r, \quad V_2/S_2=\frac{r}{2},$$

$$V_3=\frac{4}{3}\pi r^3, \quad S_3=4\pi r^2, \quad V_3/S_3=\frac{r}{3},$$

$$V_4=\tfrac{1}{2}\pi^2 r^4, \quad S_4=2\pi^2 r^3, \quad V_4/S_4=\frac{r}{4}.$$

Hence, we notice that in each case, the content is obtained from the boundary by multiplying it by the radius and dividing by the order of the containing manifold.*

179. The Hypervolume and Boundary-content of an n-dimensional Spheric :

The following method was adopted by Schoute † for calculating the hypervolume or content (*Inhalt*) and boundary-content (*Oberfläche*) of an n-dimensional spheric :

The hypervolume (I) and the boundary-content (O) of an n-spheric are connected by the relation

$$I=\frac{1}{n}.O.r, \quad i.e., \quad I=\frac{r}{n}.O,$$

where r is the radius of the spheric.

If the boundary be divided into a large number of very small polytopes (P_o),‡ and each is taken as the

* The general case for a space of n dimensions has been considered in the author's *Analytical Geometry of Hyperspaces*, Vol. I, §§ 46, 47, where it has been noticed that the power of the multiplier π is raised by unity for each alternate space.

† P. H. Schoute, *loc. cit.*, § 95, pp. 288-90.

‡ See § 190.

base of a pyramid (P_y) with vertex at the centre, then

$$\text{the content of } (P_y) = \frac{1}{n}.r.\text{ content of } (P_o)$$

$$\therefore \quad \Sigma \text{ content of } (P_y) = \frac{1}{n}.r.\ \Sigma \text{ content of } (P_o)$$

$$i.e., \quad I = \frac{1}{n}.\ O.r. \tag{179.1}$$

If $I_n(r)$ denotes the content of an n-spheric with radius r, we have

$$I_n(r) = \int_{-r}^{r} I_{n-1} \sqrt{r^2 - x^2}\ dx$$

which, by putting $x = r \sin\phi$, becomes

$$I_n(r) = r\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} I_{n-1}(r\cos\phi)\cos\phi\ d\phi.$$

But $\quad I_n(r) = a_n.r^n$, where a_n is independent of r.

$$\therefore \quad a_n = a_{n-1}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos^n\phi\ d\phi. \qquad \dots \quad (1)$$

$$\text{Now} \quad \int_0^{\frac{1}{2}\pi} \cos^{2m}\phi\ d\phi = \frac{(2m-1)(2m-3)\cdots\cdots 3.1}{2m(2m-2)\cdots\cdots\cdots 4.2}.\frac{\pi}{2}$$

$$\text{and} \quad \int_0^{\frac{1}{2}\pi} \cos^{2m-1}\phi\ d\phi = \frac{(2m-2)(2m-4)\cdots\cdots 4.2}{(2m-1)(2m-3)\cdots\cdots 5.3}$$

$$\text{Hence,} \quad a_{2m} = \frac{(2m-1)(2m-3)\cdots\cdots 3.1}{2m(2m-2)\cdots\cdots\cdots 4.2}.\pi.\ a_{2m-1}$$

$$a_{2m-1}=\frac{(2m-2)(2m-4)\cdots\cdots4.2}{(2m-1)(2m-3)\cdots\cdots5.3}\cdot 2.a_{2m-2}$$

or, $a_{2m}=\frac{\pi}{m}.a_{2m-2}$, $\quad a_{2m-1}=\frac{2\pi}{2m-1}.a_{2m-3}$,

or, generally, $$a_n=\frac{2\pi}{n}\cdot a_{n-2}. \tag{179.2}$$

Since $a_2=\pi$ and $a_1=2$, we must have

$$\mathrm{I}_{2m}(r)=\frac{\pi^m.r^{2m}}{1.2.3\cdots m} \tag{179.3}$$

$$\mathrm{I}_{2m-1}(r)=\frac{2^m.\pi^{m-1}r^{2m-1}}{1.3.5\cdots(2m-1)} \tag{179.4}$$

or, using Gamma Function,

$$\mathrm{I}_n(r)\;{}^*=\frac{[\Gamma(\frac{1}{2})]^n}{\Gamma(\frac{n}{2}+1)}r^n=\frac{\pi^{\frac{1}{2}n}}{\Gamma(\frac{n}{2}+1)}r^n. \tag{179.5}$$

For the boundary-content, we find

$$\mathrm{O}_{2m}(r)=\frac{\pi^m r^{2m-1}}{3.4.5\cdots(m-1)}$$

$$\mathrm{O}_{2m-1}(r)=\frac{2^m\pi^{m-1}r^{2m-2}}{1.3.5\cdots(2m-3)}$$

or, $$\mathrm{O}_n(r)=\frac{n[\Gamma(\frac{1}{2})]^n}{\Gamma(\frac{n}{2}+1)}r^{n-1}=\frac{2\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)}.r^{n-1}. \tag{179.6}$$

If, now, we put $\mathrm{O}_n(r)=b_n.r^{n-1}$, then, from the relation (179.1), we get

$$b_n=n.a_n.$$

And from the relation between a_n and a_{n-2}, we get

$$b_n=\frac{2\pi}{n-2}b_{n-2}. \tag{179.7}$$

* Compare Schläfli, *loc. cit.*, § 19, p. 59.

For $n=2, 3, 4, 5, 6, 7, \cdots\cdots$ we find

$$O(r)=2\pi r,\quad 4\pi r^2,\quad 2\pi^2 r^2,\quad \tfrac{8}{3}\pi^2 r^4,\quad \pi^3 r^5,\quad \tfrac{16}{15}\pi^3 r^6 \ldots$$

$$I(r)=\pi r^2,\quad \tfrac{4}{3}\pi r^3,\quad \tfrac{1}{2}\pi^2 r^4,\quad \tfrac{8}{15}\pi^2 r^5,\quad \tfrac{1}{6}\pi^3 r^6,\quad \tfrac{16}{105}\pi^3 r^7 \ldots$$

180. The Content of the spherical Sector and spherical Stratum:

If $I_n(r, h)$ denotes the content of the sector of the spheric of radius r and height h, and $I_n\ (r, h_1, h_2)$ denotes the content of the spherical stratum of the spheric bounded by two parallel hyperplanes at distances $r-h_1$ and $r-h_2$ from the centre, then for the stratum, we get

$$I_n(r,\ h_1,\ h_2)=\int_{r-h_2}^{r-h_1} I_{n-1}\ \sqrt{r^2-x^2}\ dx,$$

where $I_{n-1}\sqrt{r^2-x^2}$ denotes the content of the $(n-1)$-spheric with the radius $\sqrt{r^2-x^2}$.

Hence, we find—

$$I_{2m}\ (r, h_1, h_2)=\frac{2^m \pi^{m-1} r^{2m}}{1.3.5\cdots(2m-1)}\int_{\sin^{-1}\frac{r-h_2}{r}}^{\sin^{-1}\frac{r-h_1}{r}} \cos^{2m}\phi\ d\phi$$

$$I_{2m-1}\ (r, h_1, h_2)=\frac{\pi^{m-1}\ r^{2m-1}}{1.2.3.\cdots(m-1)}\int_{\sin^{-1}\frac{r-h_2}{r}}^{\sin^{-1}\frac{r-h_1}{r}} \cos^{2m-1}\phi\ d\phi.$$

Of these two integrals, the first is trigonometric and the second is algebraic.

When $h_1=0$ and $h_2=h$, the stratum becomes a sector.

If $h_1=k_1r$, $h_2=k_2r$, we have in the fourfold $(n=4)$

$$\begin{aligned}\mathrm{I}_4(r,\, h_1,\, h_2)=\tfrac{1}{6}\pi r^4[&(3+4k_2-2k_2{}^2)\sqrt{2k_2-k_2{}^2}\\ &-(3+4k_1-2k_1{}^2)\sqrt{2k_1-k_1{}^2}\\ &-3\{\sin^{-1}(1-k_2)-\sin^{-1}(1-k_1)\}] \quad (180.1\end{aligned}$$

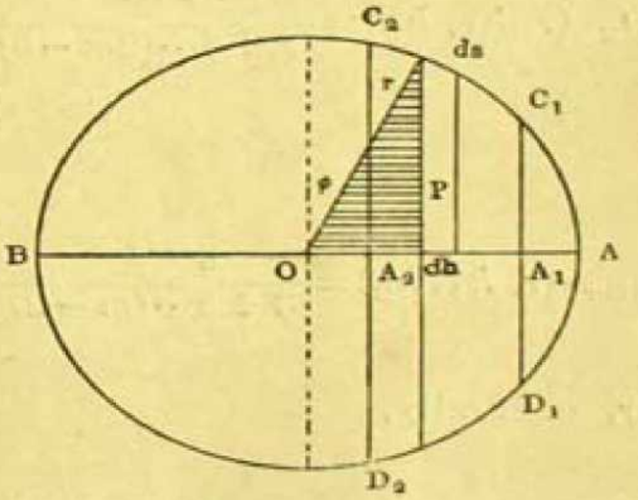
and for the sector, where $h=kr$,

$$\mathrm{I}_4(r,\, h)=\tfrac{1}{6}\pi r^4[(3+4k-2k^2)\sqrt{2k-k^2}-3\cos^{-1}(1-\mu)] \quad (180.2$$

Ex. If the hyperplane $x_1=k$ divides the spheric $x_1{}^2+x_2{}^2+x_3{}^2+x_4{}^2=a^2$ into two parts whose hypervolumes are in the ratio $2:1$, prove that

$$a^4(\pi-6\sin^{-1}\frac{k}{a})=2k\sqrt{a^2-k^2}\cdot(5a^2-2k^2).$$

181. The Boundry-content of the spherical Sector and the spherical Stratum:



Suppose a plane through the axis AB meets the stratum in the circumference of a circle with centre O and radius r, which lies between two parallel chords C_1D_1, C_2D_2.

Let $AA_1=h_1$, $AA_2=h_2$.

If now ds is an infinitely small arc of C_1C_2 and dh its projection on the axis and ρ the distance of the centre of the element from the axis, we have from two similar right-angled triangles $\rho : r=dh : ds$.

By using a known result * for the layer of the boundary described by ds, where $\frac{r_1{}^p - r_2{}^p}{r_1 - r_2}$ reduces to $p.\ \rho^{p-1}$, we get

$$O_{2m}(r,\ h_1,\ h_2) = \frac{2^m \pi^{m-1}}{1.\ 3.\ 5(\cdots 2m-3)} \int_{\text{arc AC}_1}^{\text{arc AC}_2} \rho^{2m-2}\, ds$$

$$= \frac{2^m \pi^{m-1}}{1.\ 3.\ 5 \cdots (2m-3)}\, r \int_{\text{AA}_1}^{\text{AA}_2} \rho^{2m-3}\ \ dh.$$

$$O_{2m-1}(r,\ h_1,\ h_2) = \frac{\pi^{m-1}}{3.\ 4.\ 5 \cdots (m-2)} \int_{\text{arc AC}_1}^{\text{arc AC}_2} \rho^{2m-3}\ \ ds$$

$$= \frac{\pi^{m-1}}{3.\ 4.\ 5 \cdots (m-2)}\, r \int_{\text{AA}_1}^{\text{AA}_2} \rho^{2m-4}\ \ dh.$$

Putting $\rho = r \cos\phi$, we get

$$O_{2m}(r,\ h_1,\ h_2) = \frac{2^m\ \pi^{m-1}}{1.3.5 \cdots (2m-3)}\, r^{2m-1} \int_{\cos^{-1}(1-\frac{h_1}{r})}^{\cos^{-1}(1-\frac{h_2}{r})} \sin^{2m-2}\phi\ d\phi \quad (181.1$$

$$O_{2m-1}(r,\ h_1, h_2) = \frac{\pi^{m-1}}{3\ 4.5 \cdots (m-2)}\, r^{2m-2} \int_{\cos^{-1}(1-\frac{h_1}{r})}^{\cos^{-1}(1-\frac{h_2}{r})} \sin^{2m-3}\ \phi\ d\phi \quad (181.2$$

or, generally,

$$O_n(r,\ h_1,\ h_2) = (n-1) \frac{[\Gamma(\frac{1}{2})]^{n-1}}{\Gamma(\frac{1}{2}n)} r^{n-1} \int_{\cos^{-1}(1-\frac{h_1}{r})}^{\cos^{-1}(1-\frac{h_2}{r})} \sin^{n-2}\ \phi\ d\phi$$

* P. H. Schoute, *loc. cit.*, § 10, N. 103, p. 304.

In the case of the fourfold, therefore, putting $n=4$, we get

$$O_4(r, h_1, h_2)=4\pi r^3\int_{\cos^{-1}(1-\frac{h_1}{r})}^{\cos^{-1}(1-\frac{h_2}{r})} \sin^2\phi \, d\phi \tag{181·3}$$

182. Content of Varieties of Revolution :

There are two species of varieties of revolution in the fourfold—one generated by the rotation of a surface around an axis-plane and the other generated by the rotation of a plane curve about a line.

If $z^2=f(x, y)$ is the generating surface, the equation of the variety of revolution is (§ 160)

$$w^2+z^2=f(x, y)$$

$\therefore$ The hypervolume V of the variety of revolution of the first species is given by the formula $V=\pi\iint z^2 dxdy$. (1)

If ζ be the distance of the centroid of the generating surface from the axis-plane (x, y) of rotation, then *

$$\zeta\iiint dx\, dy\, dz=\iiint z\, dx\, dy\, dz.$$

Now, let V_0 be the volume of the generating surface.

Then $$V_0.\zeta=\iiint z\, dz\, dx\, dy=\tfrac{1}{2}\iint z^2\, dx\, dy \quad \ldots \quad (2)$$

$$=\frac{1}{2\pi}\,V.$$

$$\therefore \qquad 2\pi\,\zeta.V_0=V. \tag{182·1}$$

This result can, therefore, be expressed in the following form, namely, *the hypervolume of a variety of revolution of the first species is equal to the volume of the generating surface multiplied by the length of the path described by the centroid of this volume.*

* B. Williamson, *An Elementary Treatise on the Integral Calculus*, p. 263.

Again, if S be the surface-area of the generating surface, and Z be the distance of the centroid of the boundary from the axis-plane, then

$$\text{S. Z} = \int z \, d\text{S}$$

$$\therefore \qquad 2\pi \text{Z.S} = 2\pi \int z \, d\text{S}$$

= the boundary-content of the variety of revolution. (182.2

Hence, *The boundary-content of a variety of revolution of the first species is equal to the surface-area of the generating surface multiplied by the length of the path described by the centroid of this surface.*

Note.—These are extensions of Guldin's Theorems, which were originally enunciated by Pappus. See Walton's *Mechanical Problems* (third ed.), p. 42.

183. Content of a Variety of Revolution of the Second Species:

Suppose the variety is generated by the rotation of the curve

$$y^2 = f(x)$$

about the axis of x.

The sections by the hyperplanes at right angles to the axis are spheres. Consider the section, of small thickness dx, by the hyperplane at a distance x from the origin. The content of this hyperspherical shell is $\frac{4}{3}\pi y^3 dx$.

$$\therefore \text{ The content } \text{V} = \frac{4}{3}\pi \int_a^\beta y^3 dx = \frac{4}{3}\pi \int_a^\beta \{f(x)\}^{\frac{3}{2}} dx \qquad (183{\cdot}1$$

The boundary-content of the shell is given by $4\pi y^2 dx$.

$$\therefore \qquad \text{S} = 4\pi \int_a^\beta y^2 dx = 4\pi \int_a^\beta f(x) dx \qquad \ldots \quad (183{\cdot}2$$

184. Parametric Representation :

The implicit form of the equation of a hypersurface, however convenient for discussion of topographical properties, is less suitable for discussion of intrinsic properties of curves and surfaces organically associated with the hypersurface. A parametric representation, therefore, seems desirable for the purpose. Since the variables are connected by only one relation, there are only three degrees of freedom of motion left to the corresponding points.

Consequently, three independent parameters are needed for full expression of the range of variation of the point in the fourfold. Denoting these parameters by p, q, r, we may express the co-ordinates of any point on a hypersurface in the form :—

$$x=x(p, q, r), \quad y=y(p, q, r), \quad z=z(p, q, r), \quad w=w(p, q, r)$$

subject to the condition that there exists only one relation, free from the parameters, among the variables.

If any one of the parameters be given, the co-ordinates are now functions of only two parameters and will, therefore, refer to a surface on the hypersurface. We shall have thus three families of parametric surfaces, corresponding to constant values of the three parameters.

Similarly, if two of the parameters are given constant, there will be a family of parametric curves defined by the variable parameter. Thus, again, there will be three different systems of curves defined respectively for p variable, q variable and r variable.

The curves p and q are parametric on the surface given by r constant, and so on.

The three parametric surfaces P, Q and R meet at a point O, whose position may then be considered as determined by the values of the parameters p, q and r, and these latter may then be defined as *curvilinear* co-ordinates of P.

185. Linear Element:

The elementary distance between two neighbouring points (x, y, z, w) and $(x+dx, \ y+dy, \ z+dz, \ w+dw)$ denoted by ds may be called the *linear element* of the hypersurface.

We have thus—

$$ds^2 = \Sigma\{(x+dx)-x\}^2$$

$$= dx^2+dy^2+dz^2+dw^2. \qquad \ldots \quad (1)$$

But since x, y, z, w are functions of the parameters p, q, r, we have

$$\left.\begin{aligned} dx &= \frac{\partial x}{\partial p}.dp+\frac{\partial x}{\partial q}.dq+\frac{\partial x}{\partial r}.dr \\ dy &= \frac{\partial y}{\partial p}.dp+\frac{\partial y}{\partial q}.dq+\frac{\partial y}{\partial r}.dr \\ dz &= \frac{\partial z}{\partial p}.dp+\frac{\partial z}{\partial q}.dq+\frac{\partial z}{\partial r}.dr \\ dw &= \frac{\partial w}{\partial p}.dp+\frac{\partial w}{\partial q}.dq+\frac{\partial w}{\partial r}.dr \end{aligned}\right\} \qquad \ldots \quad (2)$$

If, now, we put

$$A \equiv \Sigma\left(\frac{\partial x}{\partial p}\right)^2, \quad B \equiv \Sigma\left(\frac{\partial x}{\partial q}\right)^2, \quad C \equiv \Sigma\left(\frac{\partial x}{\partial r}\right)^2,$$

$$F \equiv \Sigma\frac{\partial x}{\partial q}.\frac{\partial x}{\partial r} \qquad G \equiv \Sigma\frac{\partial x}{\partial r}.\frac{\partial x}{\partial p}, \quad H \equiv \Sigma\frac{\partial x}{\partial p}.\frac{\partial x}{\partial q},$$

the expression for ds^2 now reduces to

$$ds^2 = Adp^2 + Bdq^2 + Cdr^2 + 2\ Fdqdr + 2Gdrdp + 2Hdpdq$$
$$= (A,\ B,\ C,\ F,\ G,\ H \between dp,\ dq,\ dr)^2 \qquad (185.1$$

From this we at once deduce an identical relation

$$(A,\ B,\ C,\ F,\ G,\ H \between p',\ q',\ r')^2 = 1, \qquad (185.2$$

$$\text{where} \quad p' = \frac{dp}{ds}, \quad q' = \frac{dq}{ds}, \quad r' = \frac{dr}{ds}.$$

The magnitudes A, B, C, F, G, H are defined to be the *fundamental magnitudes* of the first order after Gauss.*

186. The magnitudes A, B, C, F, G, H have important applications in the differential geometry of hypersurfaces and are independent of the particular selection or orientation of the orthogonal frame.

If we apply a transformation (111.1), making use of the notation

$$x_p = \frac{\partial x}{\partial p} = x_1, \quad x_q = \frac{\partial x}{\partial q} = x_2, \quad x_r = \frac{\partial x}{\partial r} = x_3, \text{ etc.},$$

we obtain, in virtue of relations (111.2)

$$A' \equiv \Sigma x'_p{}^2 = \Sigma(l_1 x_p + m_1 y_p + n_1 z_p + p_1 w_p)^2$$
$$= x_p^2 .\ \Sigma l_1^2 + y_p^2 \Sigma m_1^2 + z_p^2 \Sigma n_1^2 + w_p^2 \Sigma p_1^2$$
$$+ 2y_p z_p \Sigma m_1 n_1 + 2z_p x_p \Sigma n_1 l_1 + 2x_p y_q \Sigma l_1 m_1$$
$$+ 2x_p w_p \Sigma l_1 p_1 + 2y_p w_p \Sigma m_1 p_1 + 2z_p w_p \Sigma n_1 p_1$$
$$= x_p{}^2 + y_p{}^2 + z_p{}^2 + w_p{}^2 \equiv A.$$

Similarly,

$$F' \equiv \Sigma x'_q . x'_r = x_q x_r + y_q y_r + z_q z_r + w_q w_r$$
$$\ldots \qquad = \Sigma x_q x_r \equiv F, \text{ and so on.}$$

* Gauss, *Disquisitiones generales circa superficies curvas*, Coll. Works, Vol. IV., pp. 219-58.

187. The Normal to a Hypersurface:

The three parametric surfaces defined by $p=const.$ (P), $q=const.$ (Q) and $r=const.$ (R) intersect at a point O on the hypersurface, and they intersect two by two along the three parametric curves OU (p variable), OV (q variable) and OW (r variable). The direction-cosines * of the tangents at O to these three curves respectively may be taken as proportional to

$$x_p, y_p, z_p, w_p; \quad x_q, y_q, z_q, w_q; \quad x_r, y_r, z_r, w_r.$$

If, now, (L, M, N, P) be the direction-cosines of the normal OT to the hypersurface at O, the conditions of orthogonality of OT with OU, OV, OW give

$$\left.\begin{aligned} \mathrm{L}x_p+\mathrm{M}y_p+\mathrm{N}z_p+\mathrm{P}w_p&=0\\ \mathrm{L}x_q+\mathrm{M}y_q+\mathrm{N}z_q+\mathrm{P}w_q&=0\\ \mathrm{L}x_r+\mathrm{M}y_r+\mathrm{N}z_r+\mathrm{P}w_r&=0 \end{aligned}\right\} \quad \ldots \quad (1)$$

Ordinary method of solution, then, gives

$$\frac{\mathrm{L}}{\alpha}=\frac{\mathrm{M}}{\beta}=\frac{\mathrm{N}}{\gamma}=\frac{\mathrm{P}}{\delta}=\frac{1}{\sqrt{\Sigma\alpha^2}} \quad \ldots \quad (2)$$

where

$$\alpha\equiv\frac{\partial(w,y,z)}{\partial(p,q,r)},\ \beta\equiv\frac{\partial(z,w,x)}{\partial(p,q,r)},\ \gamma\equiv\frac{\partial(x,y,w)}{\partial(p,q,r)},\ \delta\equiv\frac{\partial(x,y,z)}{\partial(p,q,r)}.$$

We get, on simplification,

$$\begin{aligned}\alpha^2+\beta^2+\gamma^2+\delta^2&=\mathrm{ABC}+2\mathrm{FGH}-\mathrm{AF}^2-\mathrm{BG}^2-\mathrm{CH}^2\\ &\equiv\vartheta^2 \text{ (say)}\end{aligned}$$

$$\therefore \quad \mathrm{L}/\alpha=\mathrm{M}/\beta=\mathrm{N}/\gamma=\mathrm{P}/\delta=\frac{1}{\pm\vartheta}. \quad \ldots \quad (3)$$

* To be shown later in Chapter X, § 224.

Now, taking the positive square root, the direction-cosines of the normal to the hypersurface are found to be

$$\frac{\alpha}{\vartheta},\ \frac{\beta}{\vartheta},\ \frac{\gamma}{\vartheta},\ \frac{\delta}{\vartheta}. \tag{187·1}$$

If Θ denote the trihedral (solid) angle formed by the three tangent lines, we have

$$\begin{aligned} \text{ABC}\sin\Theta &= \Sigma\begin{vmatrix} x_p & y_p & z_p \\ x_q & y_q & z_q \\ x_r & y_r & z_r \end{vmatrix}^2 \\ &= \alpha^2+\beta^2+\gamma^2+\delta^2=\vartheta^2 \end{aligned}$$

whence,
$$\sin\Theta=\frac{\vartheta^2}{\text{ABC}}. \tag{187.2}$$

The orientation-cosines of the tangent plane to the parametric surface $p=$constant, which is determined by the tangent lines to OV, OW, are given by

$$\left\|\begin{matrix} x_q & y_q & z_q & w_q \\ x_r & y_r & z_r & w_r \end{matrix}\right\|$$

If, then,

$$a_1\equiv\frac{\partial(y,z)}{\partial(q,r)},\quad b_1\equiv\frac{\partial(z,x)}{\partial(q,r)},\quad c_1\equiv\frac{\partial(x,y)}{\partial(q,r)},$$

$$f_1\equiv\frac{\partial(x,w)}{\partial(q,r)},\quad g_1\equiv\frac{\partial(y,w)}{\partial(q,r)},\quad h_1\equiv\frac{\partial(z,w)}{\partial(q,r)},$$

we obtain

$$\begin{aligned} a_1{}^2+b_1{}^2+c_1{}^2+f_1{}^2+g_1{}^2+h_1{}^2 &= \Sigma(x_qy_r-x_ry_q)^2 \\ &= \Sigma x_q{}^2.\Sigma x_r{}^2-(\Sigma x_qx_r)^2 \\ &= \text{BC}-\text{F}^2 \qquad \ldots \quad (4) \end{aligned}$$

Also
$$a_1f_1+b_1g_1+c_1h_1=0. \qquad \ldots \quad (5)$$

$\therefore$ The orientation-cosines of the tangent plane to the surface $p = const.$ are proportional to $a_1, b_1, c_1, f_1, g_1, h_1$, *i.e.*, they are respectively

$$\left.\begin{array}{l} \dfrac{a_1}{\sqrt{BC-F^2}}, \dfrac{b_1}{\sqrt{BC-F^2}}, \dfrac{c_1}{\sqrt{BC-F^2}}, \\ \dfrac{f_1}{\sqrt{BC-F^2}}, \dfrac{g_1}{\sqrt{BC-F^2}}, \dfrac{h_1}{\sqrt{BC-F^2}}. \end{array}\right\} \qquad (187.3$$

Similarly, the orientation-cosines of the tangent planes to the two parametric surfaces $q = const.$ and $r = const.$ are respectively

$$\frac{a_2}{\sqrt{CA-G^2}}, \quad \frac{b_2}{\sqrt{CA-G^2}}, \quad \frac{c_2}{\sqrt{CA-G^2}}, \text{ etc.} \qquad (187.4$$

$$\frac{a_3}{\sqrt{AB-H^2}}, \quad \frac{b_3}{\sqrt{AB-H^2}}, \quad \frac{c_3}{\sqrt{AB-H^2}}, \text{ etc.} \qquad (187.5$$

where $a_2 \equiv \dfrac{\partial(y,z)}{\partial(p,r)}$, etc., $\qquad a_3 \equiv \dfrac{\partial(y,z)}{\partial(p,q)}$, etc.

The angles $\theta_1, \theta_2, \theta_3$ between the three tangent lines are given by

$$\left.\begin{array}{l} \cos\theta_1 = \dfrac{\Sigma x_q x_r}{\sqrt{\Sigma x_q^2}\sqrt{x_r^2}} = \dfrac{F}{\sqrt{BC}} \\ \cos\theta_2 = \dfrac{\Sigma x_r x_p}{\sqrt{\Sigma x_r^2}\sqrt{x_p^2}} = \dfrac{G}{\sqrt{CA}} \\ \cos\theta_3 = \dfrac{\Sigma x_p x_q}{\sqrt{\Sigma x_p^2}\sqrt{\Sigma x_q}} = \dfrac{H}{\sqrt{AB}} \end{array}\right\} \qquad (187\cdot6$$

The conditions for the three tangent lines to be mutually orthogonal are $F = G = H = 0$.

Let ϕ_1, ϕ_2, ϕ_3 be the dihedral angles between the three tangent planes. Then the angle ϕ_1 between the planes (187.4) and (187.5) is given by (§ 91)

$$\sin \theta_2 \sin \theta_3 \cos \phi_1 = \frac{\Sigma a_2 a_3}{\sqrt{CA-G^2} \cdot \sqrt{AB-H^2}}$$

$$\therefore \quad \cos \phi_1 = \frac{A\sqrt{BC}\ \Sigma a_2 a_3}{(CA-G^2)(AB-H^2)}$$

$$= \frac{A\sqrt{BC}\ (GH-AF)}{(CA-G^2)(AB-H^2)} \qquad (187.7$$

Similarly, $$\cos \phi_2 = \frac{B\sqrt{AC}\ (FH-BG)}{(BC-F^2)(AB-H^2)}$$

$$\cos \phi_3 = \frac{C\sqrt{AB}(FG-CH)}{(BC-F^2)(CA-G^2)}$$

If these three tangent planes are mutually orthogonal, we have

$$\cos \phi_1 = \cos \phi_2 = \cos \phi_3 = 0,$$

whence, $GH-AF=FH-BG=FG-CH=0$,

i. e., $$FGH = AF^2 = BG^2 = CH^2. \qquad (187.8$$

This is then *a necessary condition that the parametric surfaces at any point on a hypersurface form an orthogonal system.* This is an extension of Dupin's theorem in the ordinary space.*

188. Differential Form of a Hypersurface:

Suppose the hypersurface is defined by the equation

$$\phi(x, y, z, w) = 0 \qquad \ldots \quad (1)$$

and let L, M, N, P be the direction-cosines of the normal at any point (x, y, z, w).

* Salmon, *Geometry of Three Dimensions*, § 304, p. 308.

When a point $(x+dx,\ y+dy,\ z+dz,\ w+dw)$, contiguous to $(x,\ y,\ z,\ w)$, in the direction $dx,\ dy,\ dz,\ dw$ lies in it, the equation

$$\frac{\partial \phi}{\partial x}\,dx+\frac{\partial \phi}{\partial y}\,dy+\frac{\partial \phi}{\partial z}\,dz+\frac{\partial \phi}{\partial w}\,dw=0 \quad \ldots \quad (2)$$

is satisfied.

The equations of the line are given by

$$\frac{X-x}{dx}=\frac{Y-y}{dy}=\frac{Z-z}{dz}=\frac{W-w}{dw}.$$

Hence, the co-ordinates (X, Y, Z, W) of any point on the line satisfy the relation

$$(X-x)\frac{\partial \phi}{\partial x}+(Y-y)\frac{\partial \phi}{\partial y}+(Z-z)\frac{\partial \phi}{\partial z}+(W-w)\,\frac{\partial \phi}{\partial w}=0 \ \ldots \quad (3)$$

which is the equation of the tangent hyperplane to the hypersurface (1). Hence, the direction-cosines of the normal to the hypersurface are given by

$$L : M : N : P=\frac{\partial \phi}{\partial x} : \frac{\partial \phi}{\partial y} : \frac{\partial \phi}{\partial z} : \frac{\partial \phi}{\partial w}.$$

Hence, if L, M, N, P are given appropriate functions of the variables, then the equation

$$L dx+M dy+N dz+P dw=0 \qquad \ldots \qquad (4)$$

may be taken to represent a hypersurface, *i.e.*, it is the differential equation of a hypersurface $\phi=const.$

The functions L, M, N, P must, then, be proportional to the direction-cosines of the normal to the hypersurface, and we must have

$$\frac{\partial \phi}{\partial x}=\mu L,\ \ \frac{\partial \phi}{\partial y}=\mu M,\ \ \frac{\partial \phi}{\partial z}=\mu N,\ \ \ \frac{\partial \phi}{\partial w}=\mu P,$$

where μ is a factor of proportionality. We have, then,

$$\frac{\partial(\mu L)}{\partial y}=\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial(\mu M)}{\partial x}$$

$$\text{or, } \mu\frac{\partial L}{\partial y}+L\frac{\partial\mu}{\partial y}=\mu\frac{\partial M}{\partial x}+M\frac{\partial\mu}{\partial x}$$

$$\text{or, } \mu\left(\frac{\partial L}{\partial y}-\frac{\partial M}{\partial x}\right)=M\frac{\partial\mu}{\partial x}-L\frac{\partial\mu}{\partial y} \quad \ldots \quad (5)$$

$$\text{Similarly, } \mu\left(\frac{\partial M}{\partial z}-\frac{\partial N}{\partial y}\right)=N\frac{\partial\mu}{\partial y}-M\frac{\partial\mu}{\partial z} \quad \ldots \quad (6)$$

$$\mu\left(\frac{\partial N}{\partial x}-\frac{\partial L}{\partial z}\right)=L\frac{\partial\mu}{\partial z}-N\frac{\partial\mu}{\partial x} \quad \ldots \quad (7)$$

$$\mu\left(\frac{\partial N}{\partial w}-\frac{\partial P}{\partial z}\right)=P\frac{\partial\mu}{\partial z}-N\frac{\partial\mu}{\partial w} \quad \ldots \quad (8)$$

$$\mu\left(\frac{\partial L}{\partial w}-\frac{\partial P}{\partial x}\right)=P\frac{\partial\mu}{\partial x}-L\frac{\partial\mu}{\partial w} \quad \ldots \quad (9)$$

$$\mu\left(\frac{\partial M}{\partial w}-\frac{\partial P}{\partial y}\right)=P\frac{\partial\mu}{\partial y}-M\frac{\partial\mu}{\partial w} \quad \ldots \quad (10)$$

Multiplying (5) by (8), (6) by (9) and (7) by (10), and adding, we obtain

$$\left(\frac{\partial L}{\partial y}-\frac{\partial M}{\partial x}\right)\left(\frac{\partial N}{\partial w}-\frac{\partial P}{\partial z}\right)+\left(\frac{\partial M}{\partial z}-\frac{\partial N}{\partial y}\right)\left(\frac{\partial L}{\partial w}-\frac{\partial P}{\partial x}\right)$$
$$+\left(\frac{\partial N}{\partial x}-\frac{\partial L}{\partial z}\right)\left(\frac{\partial M}{\partial w}-\frac{\partial P}{\partial y}\right)=0 \quad (188.1$$

The functions L, M, N, P must, therefore, be subject to this relation, in order that the equation (4) may be integrable, and may represent a hypersurface of the form $\phi=const.$

189. Curvature of Hypersurfaces:

The notion of curvature of surfaces in the geometry of the ordinary space may be extended to the geometry of hypersurfaces in a fourfold, by using extended forms of Gauss's relations. We have already noticed three parametric families of surfaces and curves through any point of a hypersurface; among these organically associated curves and surfaces on a hypersurface, the most important and deserving of consi deration are those, which may be called *Surfaces of Curvature*, or *Lines of Curvature*, following the analogy to the lines of curvature in a surface.

These lines of curvature may be considered as curve-loci, along which normals to the consecutive tangent hyperplanes, *i.e.*, to the hypersurface itself, intersect. Surfaces of curvatures, as usual, may be considered as the surface-loci, along which consecutive *normal planes* * intersect.

The plane drawn through any tangent line and the normal to the hypersurface will intersect the latter in a plane curve. In the plane of this section, if a perpendicular be drawn from a point Q, contiguous to O, upon the tangent to the curve, it is the perpendicular drawn from Q to the tangent hyper-plane. If $\mathbf{p}$ denotes the length of this perpendicular, and Q is the point $(\xi, \eta, \zeta, \omega)$, then

$$\mathbf{p}=\mathrm{L}(\xi-x)+\mathrm{M}(\eta-y)+\mathrm{N}(\zeta-z)+\mathrm{P}(\omega-w).$$

If ds denotes the length of the arc OQ, the radius of curvature of the curve is the limiting value of the quantity $\dfrac{ds^2}{2\mathbf{p}}$, as Q approaches O, and denoting the

* To be explained in the next Chapter.

radius of curvature by ρ, we may write

$$\frac{1}{\rho} = \frac{2\mathbf{p}}{ds^2}, \quad \text{or,} \quad \frac{ds^2}{\rho} = 2\mathbf{p}.$$

The quantity $\frac{1}{\rho}$ is defined to be the *curvature* of the normal section. It may sometimes be called the *linear measure* of curvature of the hypersurface, as distinct from the *specific* or *spatial* measures.

Now, to determine the value of ρ, we consider the point $(\xi, \eta, \zeta, \omega)$ in the vicinity of (x, y, z, w), determined by the parameters $p+dp$, $q+dq$, $r+dr$.

Then,
$$\begin{aligned}
\xi - x &= \Sigma x_1 dp + \tfrac{1}{2}\{\Sigma x_{11} dp^2 + 2\Sigma x_{23} dqdr\} + \cdots \\
\eta - y &= \Sigma y_1 dp + \tfrac{1}{2}\{\Sigma y_{11} dp^2 + 2\Sigma y_{23} dqdr\} + \cdots \\
\zeta - z &= \Sigma z_1 dp + \tfrac{1}{2}\{\Sigma z_{11} dp^2 + 2\Sigma y_{23} dqdr\} + \cdots \\
\omega - w &= \Sigma w_1 dp + \tfrac{1}{2}\{\Sigma w_{11} dp^2 + 2\Sigma w_{23} dqdr\} + \cdots
\end{aligned}$$

where
$$\begin{aligned}
\Sigma x_1 dp &= x_1 dp + x_2 dq + x_3 dr \\
\Sigma x_{11} dp^2 &= x_{11} dp^2 + x_{22} dq^2 + x_{33} dr^2 \\
\Sigma x_{23} dqdr &= x_{23} dqdr + x_{31} drdp + x_{12} dpdq
\end{aligned}$$

Since $\Sigma \mathrm{L} x_1 = 0$, $\Sigma \mathrm{L} x_2 = 0$, $\Sigma \mathrm{L} x_3 = 0$, the linear terms in dp, dq, dr in the expression for $\mathbf{p}$ vanish.

Put $\Gamma_{ij} = \mathrm{L}x_{ij} + \mathrm{M}y_{ij} + \mathrm{N}z_{ij} + \mathrm{P}w_{ij}$
where $i = j = 1, 2, 3$, and $\Gamma_{ij} = \Gamma_{ji}$.

The six quantities Γ_{ij}, involving the second derivatives of the co-ordinates, may be called *the fundamental magnitudes of the second order*.

The expression for $\mathbf{p}$ now becomes

$$\begin{aligned}
\mathbf{p} = \Sigma \mathrm{L}(\xi - x) &= \tfrac{1}{2}(\Gamma_{11} dp^2 + \Gamma_{22} dq^2 + \Gamma_{33} dr^2 + 2\Gamma_{12} dpdq \\
&\qquad + 2\Gamma_{23} dqdr + 2\Gamma_{31} drdp) \\
&= \tfrac{1}{2}(\Gamma_{11}, \Gamma_{22}, \Gamma_{33}, \Gamma_{12}, \Gamma_{23}, \Gamma_{31} \between dp, dq, dr)^2.
\end{aligned}$$

We have also $(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{F}, \mathrm{G}, \mathrm{H} \between dp, dq, dr)^2 = ds^2$ (1)

$$\therefore \quad \frac{1}{\rho} = \frac{2\mathbf{p}}{ds^2} = \frac{(\Gamma_{11}, \Gamma_{22}, \Gamma_{33}, \Gamma_{12}, \Gamma_{23}, \Gamma_{31} \between dp, dq, dr)^2}{(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{F}, \mathrm{G}, \mathrm{H} \between dp, dq, dr)^2}$$

$$= \Gamma_{11} p'^2 + \Gamma_{22} q'^2 + \Gamma_{33} r'^2 + 2\Gamma_{12} p'q'$$

$$+ 2\Gamma_{23} q'r' + 2\Gamma_{31} r'p' \qquad \ldots \quad (2)$$

The quantity $\frac{1}{\rho}$ is called the *curvature of the normal section* of the hypersurface.

Writing $p'\sqrt{\mathrm{A}} = \frac{\mathrm{X}}{\mathrm{R}}$, $q'\sqrt{\mathrm{B}} = \frac{\mathrm{Y}}{\mathrm{R}}$, $r'\sqrt{\mathrm{C}} = \frac{\mathrm{Z}}{\mathrm{R}}$, the above relation (2) reduces to

$$\frac{\mathrm{R}^2}{\rho} = \frac{\Gamma_{11}}{\mathrm{A}} \mathrm{X}^2 + \frac{\Gamma_{22}}{\mathrm{B}} \mathrm{Y}^2 + \frac{\Gamma_{33}}{\mathrm{C}} \mathrm{Z}^2 + 2\frac{\Gamma_{23}}{\mathrm{F}} \mathrm{YZ} \cos \theta_1$$

$$+ 2\frac{\Gamma_{31}}{\mathrm{G}} \mathrm{ZX} \cos \theta_2 + 2.\frac{\Gamma_{12}}{\mathrm{H}} \mathrm{XY} \cos \theta_3 \equiv \phi \quad \text{(see § 187).}$$

Making these substitutions in the permanent relation (185.2), we obtain

$\mathrm{R}^2 = \mathrm{X}^2 + \mathrm{Y}^2 + \mathrm{Z}^2 + 2\mathrm{YZ} \cos \theta_1 + 2\mathrm{ZX} \cos \theta_2 + 2\mathrm{XY} \cos \theta_3$, *i.e.*, R represents the radius vector to any point X, Y, Z on a quadric in a hyperplane.

Taking the quadric $\phi = c$, where c is a constant, we have

$$\rho = \frac{\mathrm{R}^2}{c}, \quad i.e., \ \rho \propto \mathrm{R}^2,$$

or, in other words, the *radius of curvature of the curve is proportional to the square of the radius vector of the quadric* $\phi = c$, which may be called the *Indicatrix quadric*, and $\phi = 0$ is its *asymptotic cone*.

Differentiating the relations (1) and (2), the critical equations for determining the maximum and minimum values of ρ may be written in the forms—

$$\Gamma_{11}p'+\Gamma_{12}q'+\Gamma_{13}r'=k(\mathrm{A}p'+\mathrm{H}q'+\mathrm{G}r')$$

$$\Gamma_{12}p'+\Gamma_{22}q'+\Gamma_{23}r'=k(\mathrm{H}p'+\mathrm{B}q'+\mathrm{F}r')$$

$$\Gamma_{13}p'+\Gamma_{32}q'+\Gamma_{33}r'=k(\mathrm{G}p'+\mathrm{F}q'+\mathrm{C}r')$$

where k is an undetermined multiplier. Multiplying these relations by p', q', r' respectively, and adding, we get $k=\frac{1}{\rho}$. Substituting this value for k, and then eliminating p', q', r', we get

$$\vartheta\left(\frac{1}{\rho}\right)=\begin{vmatrix} \Gamma_{11}-\frac{\mathrm{A}}{\rho} & \Gamma_{12}-\frac{\mathrm{H}}{\rho} & \Gamma_{13}-\frac{\mathrm{G}}{\rho} \\ \Gamma_{21}-\frac{\mathrm{H}}{\rho} & \Gamma_{22}-\frac{\mathrm{B}}{\rho} & \Gamma_{23}-\frac{\mathrm{F}}{\rho} \\ \Gamma_{31}-\frac{\mathrm{G}}{\rho} & \Gamma_{32}-\frac{\mathrm{F}}{\rho} & \Gamma_{33}-\frac{\mathrm{C}}{\rho} \end{vmatrix}=0 \quad (189.1$$

This is a cubic equation * in $\frac{1}{\rho}$, and therefore gives three maximum or minimum values of $\frac{1}{\rho}$.

If, now, the values of p, q, r, p', q', r' can be found, which will provide the maximum and minimum values of $1/\rho$, these values may be defined as the *principal measures* of curvature.

* Forsyth has shown that all the three roots of this equation are real. Denoting by $1/\rho_1$, $1/\rho_2$, $1/\rho_3$, the three real roots of this equation, and putting $\mathfrak{L}=\frac{1}{\rho_1}+\frac{1}{\rho_2}+\frac{1}{\rho_3}$, $\mathfrak{F}=\frac{1}{\rho_2\rho_3}+\frac{1}{\rho_3\rho_1}+\frac{1}{\rho_1\rho_2}$, and $\mathfrak{R}=\frac{1}{\rho_1\rho_2\rho_3}$, he defines $\mathfrak{L}$ the *linear measure*, $\mathfrak{F}$ the *specific measure* and $\mathfrak{R}$ the *spatial measure* of curvature of the hypersurface at the point O. Forsyth, *loc. cit.*, Vol. II, § 281, p. 39.

Thus, the principal radii of curvature can be expressed in terms of the fundamental magnitudes of the first and second orders, and it should be observed that the three roots are generally unequal, as their equality would imply other relations among the fundamental magnitudes. If these unequal roots are in order of magnitudes, *i.e.*, when

$$\rho_1 > \rho_2 > \rho_3,$$

it is to be inferred that ρ_1 provides a *maximum* value for ρ, and ρ_3 gives a *minimum* value, while ρ_2 provides merely a *stationary* value, which is neither a true maximum, nor a true minimum.

Note.—The subject of curvature properly belongs to the differential branch and is too wide in its scope. It is not possible to do full justice to the subject within the small compass of a work of introductory character. But owing to remarkable analogy to the theory in the ordinary space, an outline, at least, seems highly desirable, and is consequently given here. For further details, Forsyth's work may be consulted.

190. Linear Polytopes:

We shall conclude this chapter by considering some of the more important configurations of the fourfold, which go under the general name of *Polytopes*. The subject has been exhaustively studied by Schoute,* who designates the closed hypersurfaces already discussed as *curved polytopes*.

A polytope is defined as any portion of a manifold bounded in any manner whatsoever. If the boundary consists of linear spaces only, the polytope is called a *linear polytope*. If, however, it is bounded

* P. H. Schoute, *loc. cit.*, Vol. II, p. 1.

by curved loci, it is called a *curved polytope*. According to this definition, then, the hypersphere and other closed quadric varieties already discussed are curved polytopes.

A linear polytope is the analogue of a polygon in a plane and a polyhedron in the ordinary space.

The simplest polytope is called a *simplex*. Thus, the simplex in a plane is formed by any three non-collinear points A, B, C, and the lines joining them two by two. These lines are supposed produced to infinity in two directions, and at each vertex there are two pairs of vertical angles. The simplex ABC is, therefore, distinct from the triangle ABC, as the latter means only the limited portion of the plane bounded by the segments AB, BC, CA, and there is only one angle at each vertex.

Similarly, in the ordinary space, any four non-coplanar points A, B, C, D will determine a simplex, the six infinite lines through each pair of points, and the four infinite planes through each group of three points will enclose a simplex. Here, at each edge there are two pairs of dihedral angles, and at each vertex there are four pairs of trihedral angles, whereas the finite portion bounded by the four vertices, the six finite edges—the segments AB, BC, etc., and the four finite faces (the triangles) ABC, BCD, etc., is the tetrahedron, having only one dihedral angle along each edge and one trihedral angle at a vertex.

191. Simple Polytopes:

In a simple polytope, two and only two hyperplane boundaries meet at each plane boundary and two

and only two plane boundaries meet at each edge. There are *complex* polytopes as well, in which more than two hyperplane boundaries may meet in a plane boundary and more than two plane boundaries meet in an edge. The difference between a simple and a complex polytope may be clearly illustrated by a simple quadrilateral and a complete quadrilateral in a plane ; the former is a simple polytope, while the latter is a complex one.

192. Euler's Polyhedral Formula :

In the ordinary space, a relation * connecting the number of vertices, edges and faces of a polyhedron is given in the form

$$V+F=E+2 \tag{192.1}$$

where V, F, and E denote the number of vertices, faces and edges respectively. Proofs of the formula may be found in elementary books on solid geometry. This formula, however, can be extended to the fourfold, and the numbers of vertices, edges, faces and cells of a polyhedroid are found to be connected by a relation of the form

$$C-F+E-V=0 \tag{192.2}$$

where C, F, E and V respectively denote the number of cells, faces, edges and vertices.

* This theorem was discovered by Euler and usually goes by his name as Euler's *Polyhedral Formula*. Discussion of the different methods of proof of this theorem and its extension to geometry of higher dimensions is given by Schoute, *loc. cit.*, Vol. II, § 2, pp. 47-65.

This extended form of Euler's theorem can be easily proved by an extension of Legendre's method by considering the different varieties of angle-sums.

Take any point within the convex polyhedroid as centre and describe a hypersphere of unit radius, and draw radii from the centre to each of the angular points of the polyhedroid. If the extremities of these radii be joined by means of arcs of great circles, the hypersphere is divided into a honeycomb of spherical polyhedra. Since the relation between the angle-sums of a spherical polyhedron is the same as for a linear polyhedron, we have

$$S_1 - 2\pi\ (n-2) = S_0$$

where S_1 represents the sum of the dihedral angles of a polyhedron and S_0 denotes the polyhedral angle, and n denotes the number of plane faces.

Now, summing up for all the cells of the honeycomb, since ΣS_0 denotes all the angles of the polyhedron, ΣS_0 is equal to $4\pi \times$ the number of polyhedral angles, *i.e.*, to $4\pi V$. Similarly, ΣS_1 is equal to the sum of all the dihedral angles of all the polyhedra, *i.e.*, to $4\pi E$. Similarly, Σn is equal to the number of all the faces of all the polyhedra, *i.e.*, to 2F. Thus

$$4\pi E - 2\pi(2F - 2C) = 4\pi V$$

or, $$C - F + E - V = 0, \text{ i.e., } C + E = F + V,$$

which is the extended form of Euler's formula, and may be stated in language as follows:

In a simple polyhedroid, the number of cells plus the number of edges is equal to the number of faces plus the number of vertices.

193. Polytopes of special Forms:

There are different forms of simple polytopes in the fourfold. Some of them possess remarkable properties. We shall make here only a passing reference to some of their properties.

Parallelotope :*

A system of parallel lines passing through the points of a given polyhedron, but not lying in the hyperplane of the same, is said to form what may be called a *Parallelotope or Prismoidal Hypersurface*. The parallel lines are called the *elements*, and the elements passing through the vertices are called *edges*. The polyhedron is called the *base*, or, the *directing* polyhedron.

A *hyperprism* consists of that portion of a prismoidal hypersurface which lies between two parallel directing polyhedra, together with those polyhedra themselves and their interiors. The distance between the two polyhedra is called its *height*.

A hyperprism is a right hyperprism, when the edges are orthogonal to the hyperplanes of the bases. If, again, the bases are the interiors of regular polyhedra, the hyperprism is said to be *regular*.

If the base of a hyperprism is a polytope of content C, and h is its height, the content V of the hyperprism is given by

$$V = C.h. \tag{193.1}$$

If, however, the axis is inclined at an angle ω with the normal to the base, and a is the length of the axis, $h = a \cos \omega$.

$$\therefore \quad V = C.a \cos \omega \tag{193.2}$$

* Schläfli uses the term *Paralleloschem*, § 5.

A parallelotope is, therefore, a hyperprism whose bases are the interiors of parallelopipeds. It has four pairs of opposite, equal, parallel parallelopipeds, whose interiors are called *cells*, and the interiors of any pair can be taken as bases. There are four sets of eight parallel edges, each set joining the vertices of two opposite cells, which become *edges*, when these cells are taken as bases. A right parallelotope, whose base is a rectangular parallelopiped, may be called a *rectangular* parallelotope The edges meeting at any vertex form an orthogonal system. (Fig., § 25.) The points (O, P′), (B, D′), (C, M′) are pairs of opposite vertices.

In a parallelotope, if the edges meeting in a vertex O are taken as the axes of an oblique system, and a, b, c, d are the lengths of these edges, the co-ordinates of opposite vertices are $O(0, 0, 0, 0)$, $P'(a, b, c, d)$; $B(0, b, 0, 0)$, $D'(a, 0, c, d)$; $C(0, 0, c, 0)$, $M'(a, b, 0, d)$, etc. Whence it is at once seen that $(\frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}c, \frac{1}{2}d)$ are the co-ordinates of the middle point of the diagonals joining these pairs of opposite vertices.

Thus, *the diagonals of a parallelotope bisect one another at a common point, which may be called a centre of symmetry for the parallelotope.*

Also, referring to the figure of § 25, we have

$$\begin{aligned} OP^2 &= OA^2 + AM^2 + MN^2 + NP^2 \\ &= a^2 + b^2 + c^2 + d^2 \end{aligned} \tag{193.3}$$

Thus, *the square of the length of a diagonal of a rectangular parallelotope is equal to the sum of the squares of its four dimensions.*

The content of the right parallelotope $= abcd$. (193.4

If, however, $a=b=c=d$, the rectangular parallelotope has its edges all equal, and is called a *hypercube*. A hypercube has, then, as its base the interior of a cube and its altitude is equal to the edge of the cube. It is, therefore, a *regular* polyhedroid. It has *eight* equal cubical cells, *twenty-four* equal faces, each a common face of two cubes, *thirty-two* equal edges, and *sixteen* vertices. The four diagonals meet in one point, forming another rectangular frame.

Here, the length of a diagonal $=\sqrt{a^2+b^2+c^2+d^2}$

$$=\sqrt{4a^2}=2a. \qquad (193.5$$

Hence, *the length of a diagonal of a hypercube is equal to twice its edge.*

$$\text{The content of a hypercube}=a^4. \qquad (193.6$$

194. The Hyperpyramid:

A hyperpyramid is formed by joining all points on the boundary of a convex polytope (a pyramid) of three dimensions as base to a fixed point O (the vertex), not lying in the hyperplane of the base. This, then, is bounded by the base and four other pyramids of three dimensions, and is called a *pentahedroid*.

A hyperplane section of a hyperpyramid, parallel to the base, is a polytope (pyramid) similar to the base. Hence, by a series of hyperplanes parallel to the base, a hyperpyramid may be divided into slabs or thin prisms of small thickness dx. The content of the section is proportional to the cube of its distance from the vertex.

Hence, if C denotes the content of the base, x its distance from the vertex and h the altitude,

the content V * of the pyramid is given by

$$V = \int_0^h \frac{x^3}{h^3} . C dx$$

$$= \tfrac{1}{4} C . h, \qquad (194.1$$

i.e., the content of a hyperpyramid is equal to one-fourth of the content of a prism with the same base and altitude.

Note.—The content of a pentahedroid has already been expressed in terms of the lengths of the edges in § 29. If A and B be the volumes of the bases, and h the altitude of a frustum of a pyramid, its content is equal to the difference of the contents of the two hyperpyramids with the same base and with the same vertex.

Thus, the content required $= \tfrac{1}{4}Ah_1 - \tfrac{1}{4}Bh_2$, where h_1 and h_2 are the distances of the two bases from the vertex.

195. Regular Polyhedroids :

A regular polyhedroid, as already defined, is a figure consisting of equal regular polyhedrons together with their interiors, the polyhedrons being joined by their faces so as to include a portion of the fourfold, and the hyperplane-angles formed at the faces being all equal to one another.

(1) A *pentahedroid* has 5 vertices, 10 edges, 10 faces and 5 cells. If a regular pentahedroid is inscribed in a hypersphere and radii are drawn at right angles to the cells, they meet the hypersphere in five points. These are again the vertices of a regular pentahedroid, *symmetrically* situated to the first with respect to the centre, and is, therefore, equal to it.

The co-ordinates of the vertices may be obtained, by taking one of the cells (tetrahedra) for the co-ordinate

* L. Schläfli, *loc. cit.*, § 7, p. 15.

hyperplane xyz in such a manner, that, taking the centroid as origin, its vertices are respectively the points A (1, −1, 1, 0), B (−1, 1, 1, 0), C (1, 1, −1, 0) and D (−1, −1, −1, 0). Take the line joining the origin to the opposite vertex of the pentahedroid as the axis of w, so that the fifth vertex is the point E (0, 0, 0, a).

Since the pentahedroid is regular, its edges are all equal, and we have

$AE^2 = AB^2$, which gives $a^2 = 5$, *i.e.*, $a = \sqrt{5}$. (195.1

The co-ordinates of the centroid of the pentahedroid are, then, $(0, 0, 0, \frac{1}{5}\sqrt{5})$

If, now, we transform to parallel axes through the centroid of the pentahedroid, the five vertices become—

$$A(1, -1, 1, -\tfrac{1}{\sqrt{5}}),\ B(-1, 1, 1, -\tfrac{1}{\sqrt{5}}),\ C(1, 1, -1, -\tfrac{1}{\sqrt{5}}),$$

$$D(-1, -1, -1, -\tfrac{1}{\sqrt{5}}) \quad \text{and} \quad E(0, 0, 0, \tfrac{4}{\sqrt{5}})$$

The length of each edge $= 2\sqrt{2}$. (195.2

(2) The *hypercube*, or a regular octahedroid, has 16 vertices, 32 edges, 24 faces and 8 cells. It may be generated by the motion of a cube in a direction at right angles to its hyperplane, through a distance equal to the edge of the cube. The centre of the hypercube is equidistant from the sixteen vertices, and a diagonal is twice as long as its edge (§ 193).

The co-ordinates of the 16 vertices can be obtained by taking the origin at the centre and the axes parallel to the edges.

The co-ordinates of the 16 vertices are—

A(−1, 1, 1, 1) ; E(1, −1, −1, 1) : A′(1, −1, −1, −1) ; E′(−1, 1, 1, −1);
B(1, −1, 1, 1) : F(−1, 1, −1, 1); B′(−1, 1, −1, −1); F′(1, −1, 1, −1) ;
C(1, 1, −1, 1) : G(−1, −1, 1, 1); C′(−1, −1, 1, −1); G′(1, 1, −1, −1) ;
D(−1, −1, −1, 1) ; H(1, 1, 1, 1); D′(1, 1, 1, −1) ; H′(−1, −1, −1, −1);

AA′, BB′, CC′, etc., are the eight diagonals. The length of an edge = 2, and that of a diagonal = 4 = twice the edge.

(3) The *hexadekahedroid*, or 16-hedroid, has 8 vertices, 24 edges, 32 faces and 16 cells. Eight points, taken at a given distance from the origin on both directions of each of the four axes of an orthogonal frame, are the vertices of a regular polyhedroid, which has four diagonals along the four axes. It is easily seen that the tetrahedrons, formed by each group of four points in each of the 16 compartments of the fourfold, are all congruent ; and these 16 tetrahedrons enclose a portion of the fourfold, which is, therefore, a linear polytope.

Each tetrahedron has 4 faces and each face is common between two of them.

Hence, the number of faces $= \dfrac{16 \times 4}{2} = 32.$

Again, each vertex is one extremity of a diagonal, and a common extremity of six of the edges. The other extremities are the extremities of the remaining three diagonals.

$\therefore$ The number of edges $= \dfrac{8 \times 6}{2} = 24.$

If the fixed distance from the origin is taken as unity, the co-ordinates of the eight vertices are easily found to be—

$$(\pm 1, 0, 0, 0), \quad (0, \pm 1, 0, 0),$$
$$(0, 0, \pm 1, 0), \quad (0, 0, 0, \pm 1).$$

The length of a diagonal $= 2$, and each edge $= 2\sqrt{2}$.

(4) There is one other regular polyhedroid in the fourfold, namely, a 24-*hedroid*, which has 24 vertices, 96 edges, 96 faces and 24 cells, each cell consisting of a regular octahedron.

The vertices of a 24-hedroid are the vertices and the centres of cells of an octahedroid. Hence, their co-ordinates, with the same axes as in the octahedroid, are

$$(\pm 1, \pm 1 \pm 1, \pm 1), \quad (\pm 2, 0, 0, 0), \quad (0, \pm 2, 0, 0),$$
$$(0, 0, \pm 2, 0) \quad \text{and} \quad (0, 0, 0, \pm 2)$$

The length of an edge $= 2$.

(5) There is another regular polyhedroid, a 600-hedroid, which can be constructed in a fourfold. A net of equal regular tetrahedrons, twenty at a point, can be so constructed that their interiors will completely fill the boundary-content of a hypersphere. The number of such tetrahedrons is 600, and they determine in a fourfold a regular polyhedroid containing 600 equal regular tetrahedrons. It is called a hexakosioihedroid, or 600-hedroid. By a proper choice of axes, the co-ordinates of the vertices can be determined. There are 120 vertices, 720 edges, 1,200 faces and 600 cells; their numbers evidently satisfy the formula (192.2). For further details, see Schläfli, *loc. cit.*, § 17, pp. 42-53, and Schoute, *loc. cit.*, Bd. 2, Nr. 76, p. 235.

Note.—For details of regular polyhedroids see P. H. Schoute, *loc. cit.* See also a paper by I. Stringham, *Regular Figures in n-dimensional Space*, Am. Journal of Math., Vol. 3 (1880), pp. 1-24.

CHAPTER IX

SURFACES IN THE FOURFOLD

196. Surfaces in the Fourfold :

Two equations in four variables, which are not linear, are generally taken to represent a surface in the fourfold.

The equations—

$$\phi(x, y, z, w)=0 \quad \text{and} \quad \psi(x, y, z, w)=0$$

may be taken to represent the surface of intersection of the two hypersurfaces ϕ and ψ. There are several defects in this mode of representation of a surface ; for, the complete intersection of the hypersurfaces may consist of two or more discrete portions, which are not geometrically continuous with one another, *i.e.*, every surface in the fourfold may not be the complete intersection of two hypersurfaces.

A fundamental question, then, arises how to represent, in general, an algebraic surface in the fourfold by equations. Various methods may be suggested and, in fact, have been suggested on the lines of representation of skew curves in the ordinary space. If the parameter, used in defining a hypersurface (§ 141) are connected by a certain relation of the form $\chi(p, q, r)=0$, this represents a surface within the hypersurface, and its investigation is entirely of a different nature, integrally connected with that of the properties of the containing hypersurface.

When a surface is given in the fourfold without any reference to the containing hypersurface, all

external measurements are to be made relative to the tangential or normal spaces of various types,—lines, planes, hyperplanes, and for discussion of intrinsic, and curvature properties of surfaces, the notion of geodesics is of fundamental importance. Hence, following Gauss's method in the ordinary space, a surface will be represented, for our purpose, by means of equations which express the co-ordinates of any point in the surface in terms of two independent parameters in the forms—

$$x=x(p, q), \quad y=y(p, q), \quad z=z(p, q), \quad w=w(p, q).$$

By using this mode of representation, much of the results in Gauss theory of surfaces in the ordinary space will formally occur in the fourfold, but it must be remembered that due notice must have to be taken throughout of the containing manifold.

197. Extension of Gauss's Notation :

The above parametric representation of a surface in the fourfold is peculiarly appropriate for the discussion of the theory of curvature, and has been used by Gauss * for that purpose in the ordinary space. In order to extend his ideas to the fourfold and to proceed with the investigation in the same, the notation of Gauss should be first explained.

The surface being represented by

$$x=x(p, q), \quad y=y(p, q), \quad z=z(p, q), \quad w=w(p, q),$$

the partial differential co-efficients of x, y, z, w in

* Gauss's Memoir, *Disquisitiones circa superficies Curvas, etc.*, Coll. Works, Vol. IV, p. 219.

regard to the parameters p, q are expressed as follow :

Using the notation $\dfrac{\partial x}{\partial p}=x_1$, $\dfrac{\partial x}{\partial q}=x_2$, etc., we may write

$$dx=\frac{\partial x}{\partial p}.dp+\frac{\partial x}{\partial q}dq, \quad \text{or,} \quad dx=x_1dp+x_2dq$$

Similarly, $\quad dy=y_1dp+y_2dq, \qquad dz=z_1dp+z_2dq,$

$$dw=w_1dp+w_2dq.$$

We write
$$\begin{aligned} E&=x_1{}^2+y_1{}^2+z_1{}^2+w_1{}^2=\Sigma x_1{}^2 \\ F&=x_1x_2+y_1y_2+z_1z_2+w_1w_2=\Sigma x_1x_2 \\ G&=x_2{}^2+y_2{}^2+z_2{}^2+w_2{}^2=\Sigma x_2{}^2 \end{aligned}$$

whence, $\qquad V^2\equiv EG-F^2=\Sigma(x_1y_2-y_1x_2)^2,$

where the positive square root is taken.

The quantities E, F, G are called *fundamental magnitudes* of the first order.

The element of arc ds, measured in the surface, is given by

$$\begin{aligned} ds^2&=dx^2+dy^2+dz^2+dw^2 \\ &=\Sigma(x_1dp+x_2dq)^2 \\ &=Edp^2+2Fdpdq+Gdq^2 \qquad (197.1 \end{aligned}$$

The locus on the surface represented by $q=const.$ is a curve along which p is variable, and the locus represented by $p=const.$, is a curve along which q is variable. Hence, there are two families of curves defined by $p=const.$, and $q=const.$, which are organically related to the surface.

The element of arc along the curve $q=$constant, obtained by putting $dq=0$, is $E^{\frac{1}{2}}dp$, and the element of arc along $p=$constant, obtained by putting $dp=0$, is $G^{\frac{1}{2}}dq$.

Since the direction-cosines of a tangent line are proportional to dx, dy, dz, dw, the direction of the tangent line along the curve $q=$constant is given by $(x_1 dp,\ y_1 dp,\ z_1 dp,\ w_1 dp)$, and consequently, the direction-cosines of the tangent are—

$$x_1/\mathrm{E}^{\frac{1}{2}},\ y_1/\mathrm{E}^{\frac{1}{2}},\ z_1/\mathrm{E}^{\frac{1}{2}},\ w_1/\mathrm{E}^{\frac{1}{2}}.$$

Similarly, the direction-cosines of the tangent to the other parametric curve $p=$constant are—

$$x_2/\mathrm{G}^{\frac{1}{2}},\quad y_2/\mathrm{G}^{\frac{1}{2}},\quad z_2/\mathrm{G}^{\frac{1}{2}},\quad w_2/\mathrm{G}^{\frac{1}{2}}.$$

If, then, ω denotes the angle between the tangents to the two parametric curves, we have

$$\cos\omega=\Sigma(x_1/\mathrm{E}^{\frac{1}{2}})(x_2/\mathrm{G}^{\frac{1}{2}})=\Sigma x_1x_2/\mathrm{E}^{\frac{1}{2}}\mathrm{G}^{\frac{1}{2}}=\mathrm{F}/(\mathrm{EG})^{\frac{1}{2}} \quad (197.2$$

$$\text{Also,}\ \sin\omega=\sqrt{1-\mathrm{F}^2/(\mathrm{EG})}=\frac{\sqrt{\mathrm{EG}-\mathrm{F}^2}}{(\mathrm{EG})^{\frac{1}{2}}}=\mathrm{V}/(\mathrm{EG})^{\frac{1}{2}}.$$

Any direction on the surface is represented by

$\dfrac{dp}{ds}=p'$, $\dfrac{dq}{ds}=q'$, subject to the condition

$$\mathrm{E}p'^2+2\mathrm{F}p''q+\mathrm{G}q'^2=1 \quad (197.3$$

If θ_1 and θ_2 are the angles which this direction makes with the parametric curves, so that $\theta_1+\theta_2=\omega$, we have

$$\left.\begin{aligned}\mathrm{E}^{\frac{1}{2}}\cos\theta_1&=\mathrm{E}p'+\mathrm{F}q', & \mathrm{E}^{\frac{1}{2}}\sin\theta_1&=\mathrm{V}q'\\ \mathrm{G}^{\frac{1}{2}}\cos\theta_2&=\mathrm{F}p'+\mathrm{G}q' & \mathrm{G}^{\frac{1}{2}}\sin\theta_2&=\mathrm{V}p'\end{aligned}\right\} \quad (197.4$$

198. The Tangent Plane :

The equation

$$\mathrm{L}(\mathrm{X}-x)+\mathrm{M}(\mathrm{Y}-y)+\mathrm{N}(\mathrm{Z}-z)+\mathrm{P}(\mathrm{W}-w)=0 \quad \ldots \quad (1)$$

where L, M, N, P are constants and X, Y, Z, W are current co-ordinates, represents a hyperplane passing

through the point O $(x\ \ y, z, w)$ on the surface. If this passes through two neighbouring points on the two parametric curves, namely,

$$(x+x_1dp,\ \ y+y_1dp,\ \ z+z_1dp,\ \ w+w_1dp)$$

and $$(x+x_2dq,\ \ y+y_2dq,\ \ z+z_2dq,\ \ w+w_2dq),$$

we must have $$\mathrm{L}x_1+\mathrm{M}y_1+\mathrm{N}z_1+\mathrm{P}w_1=0 \tag{2}$$

$$\mathrm{L}x_2+\mathrm{M}y_2+\mathrm{N}z_2+\mathrm{P}w_2=0 \tag{3}$$

Equations (1), (2), (3) represent a singly infinite system of hyperplanes passing through the same plane, whose equations are obtained by eliminating L, M, N, P in the matrix form

$$\left\| \begin{matrix} \mathrm{X}-x & \mathrm{Y}-y & \mathrm{Z}-z & \mathrm{W}-w \\ x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \end{matrix} \right\| \qquad (198.1$$

This plane is called the *tangent plane* at that point, and each of the guiding hyperplanes is called the *tangent hyperplane*. The guiding lines are the tangents to the parametric curves $p=const.$, and $q=const.$

Any direction in this plane is given by (λ, μ being parameters).

$$\left.\begin{matrix} l=\lambda x_1+\mu x_2, & m=\lambda y_1+\mu y_2, \\ n=\lambda z_1+\mu z_2 & p=\lambda w_1+\mu w_2 \end{matrix}\right\} \tag{4}$$

where $\Sigma l^2=\Sigma(\lambda x_1+\mu x_2)^2=\lambda^2\mathrm{E}+2\lambda\mu\mathrm{F}+\mu^2\mathrm{G}=1.$

From equations (1), (2) and (3), we at once deduce

$$\mathrm{X}-x=\rho x_1+\tau x_2, \qquad \mathrm{Y}-y=\rho y_1+\tau y_2,$$

$$\mathrm{Z}-z=\rho z_1+\tau z_2, \qquad \mathrm{W}-w=\rho w_1+\tau w_2 \tag{5}$$

where ρ and τ are parameters.

If r_1 be the distance from O of any point on the parametric curve $q=const.$, we have

$$r_1{}^2=\Sigma(X-x)^2=\rho^2\Sigma x_1{}^2=\rho^2 E, \text{ or, } r_1=\rho E^{\frac{1}{2}}. \tag{6}$$

Similarly, if r_2 be the distance from O of any point on the curve $p=const.$, we obtain—

$$r_2{}^2=\Sigma(X-x)^2=\tau^2\Sigma x_2{}^2=\tau^2 G, \quad i.e., \quad r_2=\tau G^{\frac{1}{2}}. \tag{7}$$

If the surface is defined as the intersection of the two hypersurfaces $\phi(x, y, z, w)=0$ and $\psi(x, y, z, w)=0$, these equations identically vanish, by putting the parametric values of the co-ordinates.

$$\therefore \qquad \Sigma \frac{\partial\phi}{\partial x} x_1=0, \qquad \Sigma \frac{\partial\phi}{\partial x} x_2=0,$$

$$\Sigma \frac{\partial\psi}{\partial x} x_1=0, \qquad \Sigma \frac{\partial\psi}{\partial x} x_2=0.$$

$$\therefore \quad \Sigma \frac{\partial\phi}{\partial x}(\rho x_1+\tau x_2)=0, \qquad \Sigma \frac{\partial\psi}{\partial x}(\rho x_1+\tau x_2)=0.$$

Hence, from (5) the co-ordinates of every point in the tangent plane at the point (x, y, z, w) satisfy the two equations

$$\Sigma(X-x)\frac{\partial\phi}{\partial x}=0, \qquad \Sigma(X-x)\frac{\partial\psi}{\partial x}=0 \tag{198.2}$$

These are the equations of the tangent plane at (x, y, z, w), which, in fact, is given as the intersection of the two tangent hyperplanes to the two hypersurfaces.

199. The Normal Plane:

The plane absolutely orthogonal to the tangent plane at any point of the surface is called the *normal plane* to the surface at that point.

Its equations may be obtained from the fact, that any line in this plane is orthogonal to every direction in the tangent plane, and consequently, to the two tangents to the parametric curves through the point.

Hence, the normal plane is given by the equations

$$\vartheta \equiv \Sigma(X-x)x_1=0 \quad \text{and} \quad \Theta \equiv (X-x)x_2=0. \qquad (199.1$$

The generating hyperplane $\vartheta + k\Theta = 0$, where k is a parameter, contains the normal plane, and is called a *normal hyperplane*.

When the surface is defined by $\phi=0$, $\psi=0$, as above, the normal plane contains the normal directions of the two tangent hyperplanes, and consequently its equations may be written in the matrix form

$$\left\| \begin{matrix} X-x & Y-y & Z-z & W-w \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} & \frac{\partial\phi}{\partial w} \\ \frac{\partial\psi}{\partial x} & \frac{\partial\psi}{\partial y} & \frac{\partial\psi}{\partial z} & \frac{\partial\psi}{\partial w} \end{matrix} \right\| = 0. \qquad (199.2$$

Alternative Method :

Any direction dx, dy, dz, dw in the surface satisfies the relations

$$\Sigma \frac{\partial\phi}{\partial x} dx = 0, \qquad \Sigma \frac{\partial\psi}{\partial x} dx = 0. \qquad \ldots \qquad (1)$$

If dx, dy, dz, dw are connected by another linear relation of the form $ldx+mdy+ndz+pdw=0$, then the three equations are sufficient to determine the ratios $dx:dy:dz:dw$, *i.e.*, a fixed direction in the tangent plane.

Hence, the three directions

$$\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}, \frac{\partial\phi}{\partial w}\right), \left(\frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}, \frac{\partial\psi}{\partial z}, \frac{\partial\psi}{\partial w}\right)$$

and (l, m, n, p) lie in one and the same plane, and we must have

$$\begin{Vmatrix} l & m & n & p \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} & \frac{\partial\phi}{\partial w} \\ \frac{\partial\psi}{\partial x} & \frac{\partial\psi}{\partial y} & \frac{\partial\psi}{\partial z} & \frac{\partial\psi}{\partial w} \end{Vmatrix}=0. \quad \ldots \quad (2)$$

Hence the line—

$$X\frac{X-x}{l}=\frac{Y-y}{m}=\frac{Z-z}{n}=\frac{W-w}{p}$$

lies in the plane (199.2).

The form (199.1) can easily be obtained from (199.2). For, we have

$$X-x=\lambda\frac{\partial\phi}{\partial x}+\mu\frac{\partial\psi}{\partial x}, \quad Y-y=\lambda\frac{\partial\phi}{\partial y}+\mu\frac{\partial\psi}{\partial y},$$

$$Z-z=\lambda\frac{\partial\phi}{\partial z}+\mu\frac{\partial\psi}{\partial z}, \quad W-w=\lambda\frac{\partial\phi}{\partial w}+\mu\frac{\partial\psi}{\partial w},$$

where λ, μ are parameters.

When the surface is defined parametrically, $\phi=0$, $\psi=0$ are identically satisfied by the parametric values of x, y, z, w.

$$\therefore \quad \Sigma\frac{\partial\phi}{\partial x}x_1=0, \quad \Sigma\frac{\partial\phi}{\partial x}x_2=0, \quad \Sigma\frac{\partial\psi}{\partial x}x_1=0, \quad \Sigma\frac{\partial\psi}{\partial x}x_2=0.$$

$$\therefore \quad \Sigma(X-x)x_1=\Sigma\left(\lambda\frac{\partial\phi}{\partial x}+\mu\frac{\partial\psi}{\partial x}\right)x_1=\lambda\Sigma\frac{\partial\phi}{\partial x}x_1+\mu\frac{\partial\psi}{\partial x}x_1=0,$$

$$\Sigma(X-x)x_2=\lambda\Sigma\frac{\partial\phi}{\partial x}x_2+\mu\Sigma\frac{\partial\psi}{\partial x}x_2=0.$$

The equations (198.2) and (199.2) clearly show that the two planes are mutually absolutely orthogonal.

200. Development of a Surface about a Point :

If the variables x, y are taken as the parameters of the surface, the equations of the surface may be written as

$$x=x, \quad y=y, \quad 2z=\phi(x, y), \quad 2w=\psi(x, y). \quad \ldots \quad (1)$$

If, further, we restrict ourselves to the neighbourhood of a *regular* point of the surface as origin, then, ϕ and ψ can be expanded in ascending powers of x and y in the forms

$$2z=lx+my+ax^2+2hxy+by^2+\cdots\cdots$$
$$2w=l'x+m'y+a'x^2+2h'xy+b'y^2+\cdots\cdots$$

If, the axes are so chosen that the plane $z=w=0$ is the tangent plane at the origin, and x and y are regarded as infinitesimals, then, in all cases, near the origin, the surface may be represented in a simple form by the equations

$$\left.\begin{aligned} z&=\tfrac{1}{2}(ax^2+2hxy+by^2) \\ w&=\tfrac{1}{2}(a'x^2+2h'xy+b'y^2) \end{aligned}\right\} \quad \ldots \quad (2)$$

retaining infinitesimals up to the second order.

The plane $x=0$, $y=0$ is the normal plane at the origin on the surface. The six constants are not geometrically independent, because of the arbitrary choice of the direction of the axes in the xy plane. There are only five independent constants, which may be chosen to determine a conic in the normal plane.*

201. Normal Section of a Surface :

It is evident that the normal plane at any point O does not intersect the surface at any other point except at the point O, but any hyperplane containing this normal plane will intersect the surface in a skew curve.

* The conic has accordingly been called the *Indicatrix*. See § 212.

In fact, any tangent line together with the unique normal plane, or the tangent plane and a direction in the normal plane, will determine a hyperplane, which is defined as a *normal hyperplane*, and this will intersect the surface in a skew curve. Thus, it appears, through each tangent line there passes one and only one normal hyperplane meeting the surface in a skew curve, and consequently, each tangent line is associated with a skew curve, which may be called a *normal hyperplane-section* of the surface.

Again, any plane, drawn through the point O and meeting the tangent plane and the normal plane thereat in two lines, intersects the surface in a plane curve. The plane of this curve is evidently simply perpendicular to the tangent plane, and in consequence, it may be called a *normal plane-section* of the surface. It will be further observed that the planes through a particular tangent line and meeting the normal plane in lines all intersect the surface in plane curves, having the given tangent line as a common tangent. Thus, all these curves may be regarded as *normal plane-sections* of the surface, and are situated in the normal hyperplane associated with the particular tangent. Associated with a particular tangent line, there is, then, an infinitude of plane curves, which may be called the *normal plane-sections* of the surface, and there is a skew curve, which may be called the *normal hyperplane-section* of the surface.

There is, therefore, a remarkable difference between the properties of a surface in the ordinary space and in the fourfold. In the ordinary space, there being a unique normal direction, all the normal plane-sections have a common line of section, and the direction of the

radius of curvature is the same for all, namely, the normal direction of the surface. But in the fourfold, the direction of the radius of curvature of normal plane-sections varies from section to section, all lying in the normal plane at O.

202. Extension of Meusnier's Theorem:

The projection of the curvature of any section of a surface on the normal section is the curvature of the normal section.

This is Meusnier's Theorem in the ordinary space, and may be generalised in various ways in higher spaces.*

Let any hyperplane and a normal hyperplane through any tangent line at any point O of the surface intersect the surface in the two skew curves C and C_o respectively, which, of course, have a common tangent line at O. Let ρ and ρ_o be the radii of circular curvature of these curves. If, then, ω is the angle between the two hyperplanes, as an extension of Meusnier's theorem, it may be proved that

$$\rho = \rho_o \cos \omega.$$

Let OT be the common tangent line at O of the surface, and let P be a point contiguous to O on the normal section through OT, and Q a point contiguous to O on the oblique section through OT. Then, a sphere can be described to touch OT and to pass through the points P and Q.

* K. Kommerell, *Die Krümmung der zweidimensionalen Gebilde in ebenen Raum von vier Dimensionen*, Dissertation, § 4, Tübingen (1897); E. E. Levi, *Saggio sulla Teoria della superficie a due dimensionen immersi in un Iperpazio*, Ann. R. Scu. Norm., Pisa, Vol. 10.

The normal hyperplane intersects the hyperplane of this sphere in a plane, which contains OT and passes through the centre of the sphere, *i.e.*, the section of the sphere by this plane is a great circle. The section by the oblique hyperplane similarly is a circle with OT as tangent. These two circles are, then, ultimately the circles of curvature at O of the sections of the surface by these hyperplanes. The planes of the two circles intersect in OT, and the angle between them is the same as between the two hyperplanes, namely, ω.

Thus the normal hyperplane intersects the sphere in a great circle of radius ρ_o, while the other hyperplane meets it in a circle of radius ρ.

Hence we obtain $\rho = \rho_o \cos \omega$.*

Note.—The four points O, T, P, Q determine a hyperplane, and the portion of the surface contained in this hyperplane is three-dimensional. The curvature of the two sections can, therefore, be determined as in the ordinary space. (See § 293, Salmon's *Geometry of Three Dimensions.*)

203. Curvature of Normal Hyperplane-sections:

Since the curvature of any hyperplane-section is deducible from that of a normal hyperplane-section, it is desirable to consider the curvature of this latter section, and for this purpose, we define the surface as the intersection of the hypersurfaces

$$\phi(x, y, z, w) = 0, \qquad \psi(x, y, z, w) = 0.$$

Let ρ be the radius of curvature of the normal hyperplane-section of the surface at any point O, and r, r'

* This is an extension of Dr. Beasant's proof of Meusnier's Theorem, *Quart. Journal of Math.*, Vol. VI, p. 140. For another proof, see Kommerell's Dissertation cited before.

those of the normal hyperplane-sections of the two hypersurfaces made by hyperplanes containing the tangent plane at O. Let ω be the angle between the hyperplanes, and ϕ, $\omega-\phi$ the angles between the given hyperplane containing the tangent plane of the surface at O, and the two normal hyperplanes respectively.

Now, the curvature of the hyperplane-section of the surface is the same as that of the section of either hypersurface by the given hyperplane, since the sections have the same osculating plane.

$\therefore$ By Meusnier's Theorem,—

$$\frac{1}{r}=\frac{\cos\phi}{\rho},\quad \frac{1}{r'}=\frac{\cos(\omega-\phi)}{\rho}=\frac{\cos\omega\cos\phi}{\rho}+\frac{\sin\omega\sin\phi}{\rho}$$

$$=\frac{\cos\omega}{r}+\frac{\sin\omega\sin\phi}{\rho}$$

$$\therefore\quad \frac{1}{\rho^2}\sin^2\omega^*=\frac{1}{r^2}-2\frac{\cos\omega}{r\,r'}+\frac{1}{r'^2}. \qquad (203.1$$

If, however, the hypersurfaces intersect at right angles, $\omega=\frac{\pi}{2}$, and the formula becomes

$$\frac{1}{\rho^2}=\frac{1}{r^2}+\frac{1}{r'^2}. \qquad (203.2$$

204. Use of Canonical Form:

Let the surface be represented by the equations

$$\left.\begin{aligned} z&=\tfrac{1}{2}(ax^2+2hxy+by^2+\cdots)\\ w&=\tfrac{1}{2}(a'x^2+2h'xy+b'y^2+\cdots)\end{aligned}\right\}\quad \cdots\quad (1)$$

where the plane $z=w=0$ is taken as the tangent plane at the point O, and consequently, $x=0$, $y=0$ is the normal plane. In fact, the surface is defined as the intersection of the two hypersurfaces (1).

* This is an extension of the corresponding formula for a curve in the ordinary space.

The first of the equations (1) represents the projection of the hypersurface in the hyperplane $w=0$, while the second represents its projection on the hyperplane $z=0$.

Consider the section by the normal hyperplane

$$y-\lambda x=0, \quad \text{or,} \quad y \cos\theta - x\sin\theta=0 \qquad \dots \quad (2)$$

where θ is the angle, which the tangent line to the skew curve of section makes with the axis of x.

The sections of these hypersurfaces by the normal hyperplane (2) are evidently skew curves. Let ρ_z and ρ_w be the radii of curvature of the normal sections of the two hypersurfaces, the sections being made along the tangent plane to the surface. If ω is the angle which the hyperplane (2) makes with one normal hyperplane, then by Meusnier's Theorem, we have

$$\rho=\rho_z \cos\omega, \quad \text{and also} \quad \rho=\rho_w \sin\omega.$$

$$\therefore \quad \frac{1}{\rho^2}=\frac{1}{\rho_z^2}+\frac{1}{\rho_w^2}. \qquad (204.1$$

This also follows from the formula in the preceding article, by putting $\omega=\pi/2$.

The radius of curvature of the section of the projection-surface in the hyperplane $w=0$ by the plane $y-\lambda x=0$, $w=0$ is given by *

$$\frac{1}{\rho_z}=a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta$$

$$=\frac{a+2h\tan\theta+b\tan^2\theta}{\sec^2\theta}=\frac{a+2h\lambda+b\lambda^2}{1+\lambda^2}.$$

Similarly, $$\frac{1}{\rho_w}=\frac{a'+2h'\lambda+b'\lambda^2}{1+\lambda^2}$$

* Salmon, *Geometry of Three Dimensions*, § 289, p. 289.

$$\therefore \quad \frac{1}{\rho^2}=\left(\frac{a+2h\lambda+b\lambda^2}{1+\lambda^2}\right)^2+\left(\frac{a'+2h'\lambda+b'\lambda^2}{1+\lambda^2}\right)^2 \quad (204.2$$

Equation (204.2) shows that, in general, ρ can never be zero, nor infinite, and that the radii of curvature for all sections have the same sign.

If λ be so chosen that both the numerators on the right vanish, *i.e.*, if the resultant of

$$a+2h\lambda+b\lambda^2 \quad \text{and} \quad a'+2h'\lambda+b'\lambda^2$$

namely, $4(ab-h^2)(a'b'-h'^2)-(a'b+ab'-2hh')^2=0$ ... (3)

the corresponding radius becomes infinite. In this case, for this section, the surface is said to have *parabolic curvature* at the point, and the point is now called a *parabolic point*.

205. The principal Radii of Curvature:

There are four maximum or minimum values of the radii of curvature of all sections of the surface. For, if we put the differential co-efficient of $1/\rho^2$ with regard to λ equal to zero, these limiting values are given by

$$\frac{\partial}{\partial\lambda}\left(\frac{1}{\rho^2}\right)=0 \quad (205.1$$

In these investigations, we shall use the following notational abbreviations throughout :—

$$a+2h\lambda+b\lambda^2\equiv \mathrm{P}, \qquad a'+2h'\lambda+b'\lambda^2\equiv \mathrm{P}'$$

$$h\lambda^2+(a-b)\lambda-h\equiv \mathrm{Q}, \qquad h'\lambda^2+(a'-b')\lambda-h'\equiv \mathrm{Q}'$$

$$ab-h^2\equiv \mathrm{H}, \quad a'b'-h'^2\equiv \mathrm{H}', \quad ab'+a'b-2hh'\equiv \mathrm{T}$$

$$\mathrm{R}\equiv(a'+h'\lambda)(h+b\lambda)-(a+h\lambda)(h'+b'\lambda).$$

On simplification, the equation (205.1) then takes the form

$$\mathrm{PQ}+\mathrm{P'Q'}=0 \quad (205.2$$

This is a biquadratic giving four values of λ. Hence, at any point of a surface, there are *four* hyper-plane sections, which possess radii of curvature having maximum or minimum values corresponding to the four values of λ. These are defined as the four *principal radii of curvature* at the point, while in the case of an ordinary three-dimensional surface, there are only two principal radii, the other two being considered infinite. Hence, we conclude that, *a surface in the fourfold possesses, in general, four principal radii of curvature.*

If the surface lies wholly in the hyperplane $w=0$, we have $a'=h'=b'=0$, and the equation (205.2) reduces to

$$P.Q=(a+2h\lambda+b\lambda^2)\{h\lambda^2+(a-b)\lambda-h\}=0 \qquad (205.3$$

This determines the principal sections of the surface. The first factor, being equated to zero, makes ρ_2 infinite, and the two corresponding values of λ give the *asymptotic* directions. The second factor gives two values of λ, which determine the lines of curvature.*

Again, if the resultant $4HH'-T^2=0$, the corresponding value of λ satisfies the equation (205.2), and one of the radii is infinite. The point is *parabolic* on the surface. Hence, *at a parabolic point, there are only three finite principal radii of curvature.*

It will be observed that, in this case, the hyper-plane $y=\lambda x$ intersects the projections of the surface in the hyperplanes $z=0$ and $w=0$ each in two lines which are *asymptotic*.

Since the equation $PQ+P'Q'=0$ is also satisfied by putting $P=0$, $Q'=0$; $P'=0$, $Q=0$; or, $P=0=P'$, we

* Salmon, *Geometry of Three Dimensions*, § 301, p. 303. (Lines of curvature in its extended sense will be explained later.)

conclude that the hyperplane of a principal section meets one projection-surface in asymptotic directions, and the other in the lines of curvature, or meets both in lines of curvature.

206. Intersection of consecutive Normal Planes:

The equations of the normal plane at any point (x, y, z, w) of a surface are given by

$$\Sigma(X-x)x_1=0, \qquad \Sigma(X-x)x_2=0 \qquad \ldots \quad (1)$$

The equations of the normal plane at a neighbouring point $(x+dx,\ y+dy,\ z+dz,\ w+dw)$ are given by

$$\Sigma(X-x)x_1=Edp+Fdq, \qquad \Sigma(X-x)x_2=Fdp+Gdq \quad (2)$$

Solving these equations (1) and (2) for $(X-x)$, $(Y-y)$, $(Z-z)$ and $(W-w)$, we see that, corresponding to each direction $\frac{dp}{dq}$, there is a point $L(X, Y, Z, W)$ on the normal plane at the point $O(x, y, z, w)$ on the surface. For different values of $\frac{dp}{dq}=\lambda$, we get different normal planes at the points in the neighbourhood of O, and all these normal planes intersect the normal plane at O in points lying on a locus, which Kommerell * designates as the *Characteristic*.

We use the following abbreviated notations:

$$\Theta_{ij}\equiv\begin{vmatrix} \lambda_x & \lambda_y & \lambda_z & \lambda_w \\ x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_{ij} & y_{ij} & z_{ij} & w_{ij} \end{vmatrix},$$

* Kommerell, *Riemannsche Flächen in ebenen Raum von vier Dimensionen*, Math. Ann. Bd. 60, pp. 548-96.

$$\vartheta_x = \frac{\partial \Theta_{11}}{d\lambda_x}, \quad \vartheta_y = \frac{\partial \Theta_{11}}{\partial \lambda_y}, \quad \vartheta_z = \frac{\partial \Theta_{11}}{\partial \lambda_z}, \quad \vartheta_w = \frac{\partial \Theta_{11}}{\partial \lambda_w}$$

$$\Gamma_x = \frac{\partial \Theta_{12}}{\partial \lambda_x}, \quad \Gamma_y = \frac{\partial \Theta_{12}}{\partial \lambda_y}, \quad \Gamma_z = \frac{\partial \Theta_{12}}{\partial \lambda_z}, \quad \Gamma_w = \frac{\partial \Theta_{12}}{\partial \lambda_w}$$

$$\Lambda_x = \frac{\partial \Theta_{22}}{\partial \lambda_x}, \quad \Lambda_y = \frac{\partial \Theta_{22}}{\partial \lambda_y}, \quad \Lambda_z = \frac{\partial \Theta_{22}}{\partial \lambda_z}, \quad \Lambda_w = \frac{\partial \Theta_{22}}{\partial \lambda_w}$$

$$e = \begin{vmatrix} x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_{11} & y_{11} & z_{11} & w_{11} \\ x_{12} & y_{12} & z_{12} & w_{12} \end{vmatrix}, \quad 2f = \begin{vmatrix} x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_{11} & y_{11} & z_{11} & w_{11} \\ x_{22} & y_{22} & z_{22} & w_{22} \end{vmatrix},$$

$$g = \begin{vmatrix} x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_{12} & y_{12} & z_{12} & w_{12} \\ x_{22} & y_{22} & z_{22} & w_{22} \end{vmatrix}.$$

Solving equations (1) and (2), we get—

$$X - x = \frac{(E\Gamma_x - F\vartheta_x)dp^2 + (E\Lambda_x - G\vartheta_x)dpdq + (F\Lambda_x - G\Gamma_x)dq^2}{edp^2 + 2fdpdq + gdq^2}$$

$$Y - y = \frac{(E\Gamma_y - F\vartheta_y)dp^2 + (E\Lambda_y - G\vartheta_y)dpdq + (F\Lambda_y - G\Gamma_y)dq^2}{edp^2 + 2fdpdq + gdq^2}$$

with similar expressions for $Z - z$ and $W - w$.

Hence, $X - x$, $Y - y$, $Z - z$, $W - w$ are functions of the form

$$\frac{\alpha\lambda^2 + 2\beta\lambda + \gamma}{e\lambda^2 + 2f\lambda + g},$$

in which α, β, γ are functions of the primary magnitudes and their derivatives. Thus, the locus of

(X, Y, Z, W) may be obtained by eliminating λ between these relations, and is found to be a conic, which has been called the *Characteristic*.*

The asymptotic directions of this conic are given by

$$e\lambda^2+2f\lambda+g=0 \qquad \dots \quad (3)$$

Corresponding to these two directions, there are two directions on the surface itself, which may be called the *asymptotic directions* of the surface, and are determined by the values of λ given by (3).

The Characteristic will be a hyperbola, a parabola or an ellipse, according as the values of λ are real and distinct, equal, or imaginary, *i.e.*, according as

$$f^2-eg >, =, \text{ or } < 0.$$

The points on the surface can accordingly be classified into three distinct categories, namely, *hyperbolic*, *parabolic*, or *elliptic*, according as the Characteristic is a *hyperbola*, *a parabola*, or *an ellipse*.

207. Investigation with the canonical Form :

Suppose the approximate form of the surface near the origin is defined in the canonical form :—

$$\left.\begin{aligned} z&=\tfrac{1}{2}(ax^2+2hxy+by^2) \\ w&=\tfrac{1}{2}(a'x^2+2h'xy+b'y^2) \end{aligned}\right\} \qquad \dots \quad (1)$$

where x, y are taken as parameters. Putting

* Forsyth has obtained an expression for the equation of the Characteristic, *Geometry of Four Dimensions*, § 251. The point L has been called the *orthogonal centre* and OL the *orthogonal radius*.

$\dfrac{dy}{dx}=\lambda$, the equations of the Characteristic for the origin O on the surface are given by—

$$\left.\begin{array}{ll} X=0, & Y=0, \\ Z=\dfrac{Q'}{R}, & W=-\dfrac{Q}{R} \end{array}\right\} \qquad (207.1$$

Corresponding to any given value of λ, we obtain a point on the Characteristic, which lies in the normal plane at the origin, and there is a corresponding arc-element in which the normal hyperplane $y=\lambda x$ meets the surface. In fact, this hyperplane meets the surface along a skew curve, the osculating plane of which at O meets the tangent plane in a line, and hence lies in one and the same hyperplane. Its equations are, therefore, given by

$$Y=\lambda X, \quad \frac{Z}{W}=\frac{a+2h\lambda+b\lambda^2}{a'+2h'\lambda+b'\lambda^2}=\frac{P}{P'} \quad \dots \quad (2)$$

$$\text{or,} \quad Y=\lambda X, \quad \frac{Z}{P}=\frac{W}{P'} \quad \dots \quad (2')$$

If, now, L is the point on the Characteristic corresponding to this value of λ, the equations of the line OL are

$$X=0,\ Y=0, \quad \frac{Z}{W}=-\frac{h'\lambda^2+(a'-b')\lambda-h'}{h\lambda^2+(a-b)\lambda-h}=-\frac{Q'}{Q} \quad \dots \quad (3)$$

$$\text{or,} \quad \frac{X}{0}=\frac{Y}{0}=\frac{Z}{Q'}=\frac{W}{-Q} \quad \dots \quad (3')$$

$\therefore$ Its direction-cosines are—

$$0, \quad 0, \quad \frac{Q'}{\sqrt{Q^2+Q'^2}}, \quad \frac{-Q}{\sqrt{Q^2+Q'^2}}$$

The normals to the two hyperplanes (2′) have direction-cosines

$$\left(\frac{\lambda}{\sqrt{1+\lambda^2}},\ \frac{-1}{\sqrt{1+\lambda^2}},\ 0,\ 0\right) \text{ and } \left(0,\ 0,\ \frac{P'}{\sqrt{P^2+P'^2}},\ \frac{-P}{\sqrt{P^2+P'^2}}\right).$$

If α, β are the angles which OL makes with these two normals, and γ be the angle between the normals, we have

$$\cos\alpha=0,\quad \cos\beta=\frac{PQ+P'Q'}{\sqrt{P^2+P'^2}\sqrt{Q^2+Q'^2}},\quad \cos\gamma=0.$$

If ω be the angle between OL and the osculating plane (2′), we have (§ 73)

$$\sin\omega=\frac{PQ+P'Q'}{\sqrt{P^2+P'^2}\sqrt{Q^2+Q'^2}} \tag{207.2}$$

Hence, when OL lies in the osculating plane, $\omega=0$, and the necessary condition is that the numerator must vanish, *i.e.*,

$$PQ+P'Q'=(a+2h\lambda+b\lambda^2)(h\lambda^2+(a-b)\lambda-h)+$$
$$(a'+2h'\lambda+b'\lambda^2)(h'\lambda^2+(a'-b')\lambda-h')=0,$$

which is the same equation as (205.2).

The corresponding value of λ, therefore, gives a section of principal curvature, and we obtain the following

Theorem: *The point of intersection of consecutive normal planes in the direction of a given principal section lies in the osculating plane of the normal section through that direction.*

If, again, OL is normal to the osculating plane, $\omega=\frac{\pi}{2}$, and the necessary condition is—

$$\cos\omega=0,\ \ i.e.,\ \ (PQ+P'Q')^2=(P^2+P'^2)(Q^2+Q'^2)$$

$$\text{or}\qquad (PQ'-P'Q)^2=0 \qquad \dots \quad (4)$$

$$\text{But}\qquad PQ'-P'Q\equiv R(1+\lambda^2) \qquad \dots \quad (5)$$

∴ The condition (4) implies that

$$R=\begin{vmatrix} a'+h'\lambda & h'+b'\lambda \\ a+h\lambda & h+b\lambda \end{vmatrix}=0 \qquad \dots \quad (4')$$

The last condition shows that the denominator in the equations (207.1) of the Characteristic vanishes, *i.e.*, the values of λ give the asymptotic directions of the Characteristic. Hence, *for the asymptotic directions, OL is normal to the osculating plane*, which is analogous to the property in the ordinary space that, for the asymptotic curves, the normal to the surface is orthogonal to the osculating plane.

208. Axes of the Characteristic:

Since the asymptotic directions correspond to the values of λ given by the relation (4′), the equations of the asymptotes may be obtained by eliminating λ between

$$\begin{vmatrix} a'+h'\lambda & h'+b'\lambda \\ a+h\lambda & h+b\lambda \end{vmatrix}=0. \qquad \frac{Z}{W}=-\frac{h'\lambda^2+(a'-b')\lambda-h'}{h\lambda^2+(a-b)\lambda-h}$$

Squaring the determinant, and applying the identity (5), we obtain—

$$HP'^2-TPP'+H'P^2=0. \qquad \text{Also,}\ \frac{P'}{P}=-\frac{Z}{W} \qquad (1)$$

$$\therefore \quad H\left(-\frac{Z}{W}\right)^2 - T\left(-\frac{Z}{W}\right) + H' = 0,$$

or $$HZ^2 + TZW + H'W^2 = 0 \qquad (208.1$$

These are the equations of the asymptotes of the Characteristic. The axes of the Characteristic are the bisectors of the angles between the asymptotes (208·1) and are, therefore, given by *

$$Z^2 - 2\frac{H-H'}{T}.ZW - W^2 = 0$$

or $$TZ^2 - 2(H-H')ZW - TW^2 = 0 \qquad (208.2$$

These may be called the principal hyperplanes of curvature, since they intersect the normal plane in the axes of the Characteristic.

Again, since the value of $\frac{Z}{W}$ does not vary when λ is replaced by $-\frac{1}{\lambda}$, it follows at once that the line OL meets the Characteristic in two points L_1 and L_2, whose parameters are λ and $-\frac{1}{\lambda}$; these again are associated with two mutually orthogonal line-elements on the surface, defined by the hyperplanes

$$Y = \lambda X, \quad \lambda Y + X = 0.$$

Hence, we obtain the theorem:

The normal planes at the extremities of two mutually orthogonal line-elements, drawn through any point on the surface, meet the normal plane at the point in two points collinear with the surface-point.

* Salmon, *Conics*, § 75.

It must be noted at the same time, that the surface-point is not the centre of the Characteristic; for, in that case, when λ is replaced by $-\frac{1}{\lambda}$, the values of the expression Z^2+W^2, which determine the lengths of the radii drawn to the points, must remain unchanged, but the values obtained are different, showing that the points L_1, L_2 are not at the same distance from O. In fact, the line drawn in the normal plane through the surface-point meets the Characteristic in two real points, the surface-point lying within the conic.

209. Intersection with a Tangent Hyperplane:

The tangent hyperplane $Z=0$ intersects the normal plane at the point in a line, *i.e.*, in the axis of W. If W_1, W_2 are the two segments made by the Characteristic on this line, the corresponding parameters λ_1, λ_2 are obtained by putting $Z=0$, *i. e.*, by the two values of λ satisfying the equation

$$Q' \equiv h'\lambda^2 + (a'-b')\lambda - h' = 0 \qquad \ldots \quad (1)$$

Since $(PQ'-P'Q) \equiv R(1+\lambda^2)$, from the equations (207.1) of the Characteristic, we find—

$$\frac{1}{W} = -\frac{R}{Q} = \frac{P'Q}{(1+\lambda^2)Q} = \frac{P'}{1+\lambda^2} \qquad \ldots \quad (2)$$

Hence, the segments W_1 and W_2 are determined by the relations (1) and (2).

Now, the projection of the surface on the hyperplane $Z=0$ is given by

$$w = \tfrac{1}{2}(a'x^2 + 2h'xy + b'y^2), \quad z=0,$$

of which the axis of W is the normal.

The radius of curvature of the projection-surface is given by

$$\frac{1}{\rho}=\frac{a'+2h'\lambda+b'\lambda^2}{1+\lambda^2}=\frac{P'}{1+\lambda^2} \qquad \dots \quad (3)$$

where λ satisfies the equation (1).

Hence, we obtain $\frac{1}{\rho}=\frac{1}{W}$. (209.1

Since we may take any tangent hyperplane as the hyperplane $Z=0$, from these results we at once obtain the following

Theorem: *Any tangent hyperplane intersects the normal plane at a point of a surface in a line, meeting the Characteristic in two points, which are the principal centers of curvature, and the distances of these points from the surface-point are the principal radii of curvature of the projection of the surface on the tangent hyperplane.*

Again, through the tangent plane at any point of a surface, an infinite number of tangent hyperplanes can be drawn. The projections of the surface on these hyperplanes are an infinite number of ordinary surfaces, the normals to which at the point generate the normal plane to the given surface. The locus of the two principal centres of curvature of these projection-surfaces is the *Characteristic*. This property, then, may be used for constructing the Characteristic.

210. The Pedal of the Characteristic:

Let (Z', W') be a point on the Characteristic specified by the parameter λ. Then the equation of the

tangent line at this point is obtained in the usual manner * in the form

$$\frac{W-W'}{Z-Z'}=-\frac{a+2h\lambda+b\lambda^2}{a'+2h'\lambda+b'\lambda^2} \qquad \dots \quad (1)$$

The equation of the perpendicular drawn from the surface-point O on this tangent is

$$\frac{W}{Z}=\frac{a'+2h'\lambda+b'\lambda^2}{a+2h\lambda+b\lambda^2} \qquad \dots \quad (2)$$

The osculating plane of the normal section by the hyperplane $Y=\lambda X$ contains the centre of circular curvature, and its equations are

$$\frac{W}{Z}=\frac{a'+2h'\lambda+b'\lambda^2}{a+2h\lambda+b\lambda^2}, \qquad Y=\lambda X. \qquad \dots \quad (3)$$

The centre of curvature also lies in the normal plane. Hence, the line in which this osculating plane meets the normal plane $X=0$, $Y=0$ passes through the centre of curvature of the section. This line of intersection is evidently the line (2). Hence, the centre of circular curvature of the section lies on the perpendicular drawn from the surface-point on the tangent to the Characteristic.

If p denotes the length of the perpendicular drawn from O on the tangent (1), it can be shown, by using the identity $PQ'-P'Q\equiv(1+\lambda^2)R$, that

$$\frac{1}{p^2}=\left(\frac{a+2h\lambda+b\lambda^2}{1+\lambda^2}\right)^2+\left(\frac{a'+2h'\lambda+b'\lambda^2}{1+\lambda^2}\right)^2 \dots \quad (4)$$

Comparing this with (204.2), it is at once evident that the length of this perpendicular is equal to the

* *Theory of Plane Curves*, Vol. 1, § 235

radius of curvature of the normal section through the direction λ.

Hence, *the foot of the perpendicular drawn from the surface-point O on the tangent to the Characteristic is the centre of curvature of the corresponding normal section.*

But the locus of the foot of the perpendicular is the *pedal* of the Characteristic with the surface-point as origin. Hence, we obtain the following

Theorem : *The locus of the centres of curvature of the normal sections of a surface at any point is the pedal of the Characteristic with respect to that point.*

This locus is a *bicircular quartic* * and intersects the conic in eight points ; but these points occur in *four* pairs of consecutive points, for, when the perpendicular becomes a normal to the conic, the locus touches it at the foot of the normal. Since there are four normals, the locus touches the conic in four points, at the feet of the four normals. Again, these four points of contact are the *four centres* of principal curvature. For, when the normal passes through the point (Z', W'), we obtain, from the equations of the Characteristic and (3), the relation

$$PQ + P'Q' = 0 \qquad \dots \quad (5)$$

which gives the four values of λ, corresponding to the four principal radii of curvature (§ 205).

These results may be summarised as follows :

The normal hyperplane, specified by the normal plane at any point O of the surface and a direction

* *Theory of Plane Curves*, Vol. 2, § 191.

λ through the same, intersects the surface in a skew curve Γ. The normal plane at the extremity of the direction meets the normal plane at a point L on the Characteristic. The centre of curvature of the curve Γ at O is the foot of the perpendicular drawn from O on the tangent at L to the conic. The osculating plane of Γ at O is the plane determined by the direction λ and this perpendicular. The locus of the centres of curvature of all the normal sections through different directions on the surface is the pedal of the Characteristic, which is a bicircular quartic, touching it at four points, *i.e.*, at the feet of the four normals drawn from O, and these are the four *centres of principal curvature* of the surface.

211. Particular Cases of the Characteristic:

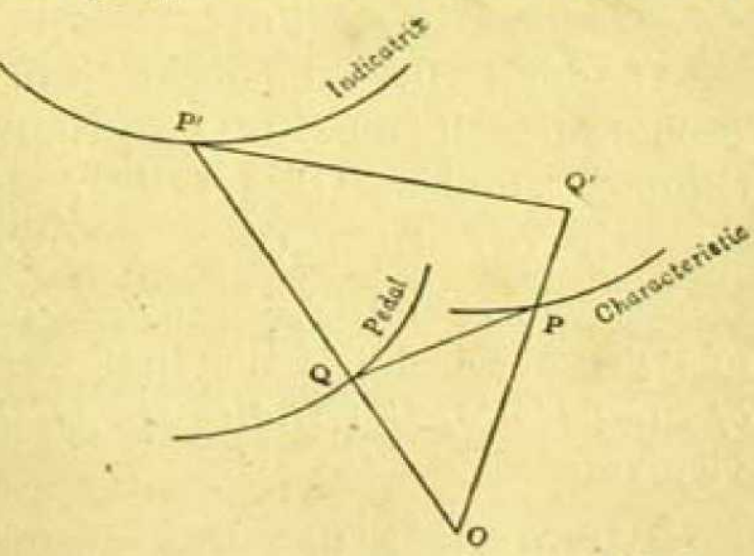
Since the surface-point lies inside the Characteristic and, from any point inside a conic, two real normals can be drawn to it, it follows that, of the four principal radii at least two are real. If the Characteristic is a parabola, one normal is infinite ; and only three finite normals can be drawn. There are, therefore, three corresponding radii and centres of principal curvature. If two radii are equal, the surface-point evidently lies on one of the axes of the Characteristic, and the two principal sections, corresponding to the other two radii, are orthogonal to each other. If three radii are equal, the Characteristic must be a circle, having the surface-point as its centre. In this case, all the four radii are equal and the projection-surfaces have all equal radii, but lying in opposite directions.

If, however, the Characteristic is a circle, but the surface-point is not its centre, the projection-surfaces have the same *specific curvature* (§ 216). In this case, there are only two normals to the circle, and consequently two real principal radii.

212. The Indicatrix :

The pedal of the Characteristic is a bicircular quartic, which touches it at the four principal centres of curvature. The bicircular quartic has a double point at the origin, namely, the surface-point O. If, now, we invert the bicircular quartic with respect to the point O, the inverse is a conic,* which has been called the *Indicatrix*. In fact, this Indicatrix is the reciprocal polar of the Characteristic, which has already been described as the locus of intersection of consecutive normal planes with the normal plane at a surface-point.

Let O be the surface-point, P any point on the Characteristic, OQ the perpendicular drawn from O on the tangent PQ. Then, Q is a centre of curvature of the surface at O. If on OQ, a point P′ be taken such that $OP'.OQ=1$, P′ is the inverse of Q. Thus, OQ is a radius of curvature and OP′ represents the measure of curvature. The locus of P for different line-elements at O is the Characteristic, and



* See my *Theory of Plane Curves*, Vol. 2, § 182. When the origin of inversion is a double point, the inverse reduces to a conic.

the locus of Q is a bicircular quartic, which is the pedal of the Characteristic with respect to O, and is the locus of the centres of curvature, while the locus of P′ is the inverse of the locus of Q with regard to the same origin and is a conic (since the origin O is a double point on the bicircular quartic) which is called the *Indicatrix,** and is really the locus of measures of curvature at O. If the Characteristic be represented by an equation of the form

$$\mathbf{a}z^2+\mathbf{b}w^2+2\mathbf{h}zw+2\mathbf{f}w+2\mathbf{g}z+\mathbf{c}=0,$$

the equation of the pedal is obtained in the form—

$$\mathbf{C}(z^2+w^2)^2-2(\mathbf{F}w+\mathbf{G}z)(z^2+w^2)+(\mathbf{A}z^2+\mathbf{B}w^2+2\mathbf{H}zw)=0$$

The inverse of this with regard to the origin † is—

$$\mathbf{A}z^2+\mathbf{B}w^2+2\mathbf{H}zw-2\mathbf{F}w-2\mathbf{G}z+\mathbf{C}=0,$$

where $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$ denote the co-factors of $\mathbf{a}, \mathbf{b}, \mathbf{c}$, etc. in the determinant

$$\begin{vmatrix} \mathbf{a} & \mathbf{h} & \mathbf{g} \\ \mathbf{h} & \mathbf{b} & \mathbf{f} \\ \mathbf{g} & \mathbf{f} & \mathbf{c} \end{vmatrix}$$

If P′Q′ is the tangent at P′ to the Indicatrix, OP produced will meet P′Q′ at right angles at Q′. From similar triangles OPQ, OP′Q,′ we have

$$OP:OQ=OP':OQ', \quad \text{or,} \quad OP.OQ'=OP'.OQ=1.$$

Hence, P and Q′ are inverse points, but the locus of Q′ is the pedal of the Indicatrix. Thus, *the inverse of the pedal of the Indicatrix is the Characteristic, and* vice versa.

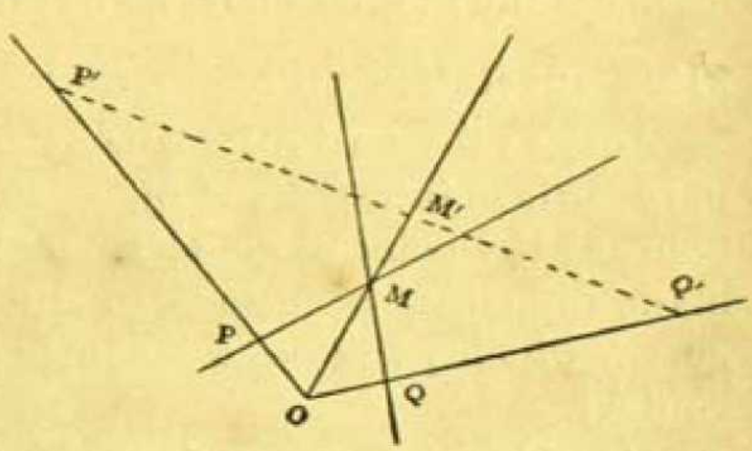
* E. B. Wilson and C. L. E. Moore, *Differential Geometry of Two-dimensional Surfaces in Hyperspace*, Proc. of the Am. Academy of Art and Sciences, Vol. 52 (1916), pp. 273-363. The authors define *the Indicatrix as the locus of geodesic* curvature in the normal plane; the treatment, however, is vectorial.

† Salmon, *Conics*, § 321.

213. Degenerate Characteristic :

The case when the Characteristic degenerates into two straight lines deserves consideration. It may be observed that in this case the Indicatrix reduces to two coincident lines. This can be easily verified by considering the equations in the preceding article. The lines drawn at right angles to the two lines constituting the Characteristic meet this *linear* Indicatrix in two points P′ and Q′, which may be called the extremities of the linear segment.

The foot M′ of the perpendicular drawn from the surface-point O to the linear segment P′Q′ is the inverse of the point of intersection M of the two lines of the Characteristic. Let PM and QM be the two lines constituting the Characteristic. OP and OQ are the perpendiculars drawn from the surface-point O on these lines.



If P′ and Q′ are inverse points of P and Q respectively, such that $OP.OP' = OQ.OQ' = 1$, then P′ and Q′ are the extremities of the linear segment, which is the Indicatrix in this case. It is easy to show that the line OM produced meets the linear segment P′Q′ at right angles at M′, such that M′ is the inverse of M, *i.e.*, $OM.OM' = 1$.

In this case, then, the points of intersection of consecutive normal planes cluster more and more closely

about the point M. Hence, the normal planes all pass through this common point M, only the two consecutive normal planes in the direction of *the axes* may be said to cut the normal plane at O in the lines PM and QM.

If, however, M is at infinity, *i.e.*, when the Characteristic reduces to two parallel straight lines, the consecutive normal planes do not meet the normal plane at O in any finite point; only those two corresponding to the axes meet the latter in two parallel lines. In this case, P′Q′ passes through O, and the surface is three-dimensional near O. Hence, when the surface is three-dimensional, normal planes do not generally intersect the normal plane at the point.

214. Involution on a Surface:

Just as the tangent plane at any point of an ordinary surface intersects it in a curve having a double point at the point of contact, so also a tangant hyperplane intersects a surface in the fourfold in a skew curve, which possesses a singular point at the point of contact.

Consider the surface defined near the origin by the equations

$$\left.\begin{aligned} z&=\tfrac{1}{2}\,(ax^2+2hxy+by^2) \\ w&=\tfrac{1}{2}\,(a'x^2+2h'xy+b'y^2) \end{aligned}\right\} \quad \ldots \quad (1)$$

Let us use the following further abbreviations:

$$f_1 \equiv ax^2+2hxy+by^2, \qquad f_2 \equiv a'x^2+2h'xy+b'y^2$$

$$\vartheta \equiv \begin{vmatrix} ax+hy & hx+by \\ a'x+h'y & h'x+b'y \end{vmatrix}$$

A tangent hyperplane denoted by $\quad z-kw=0 \quad \ldots \quad (2)$

intersects the surface (1) in a skew curve, which has a singular point at the point of contact (the origin). The tangents at this point are determined by the equation

$$f_1 - kf_2 \equiv (a - ka')\, x^2 + 2(h - kh')xy + (b - kb')y^2 = 0 \quad \ldots \quad (3)$$

and they evidently lie in the tangent plane, *i.e.*, the plane of XY.

The lines (3) evidently belong to the system in involution, determined by the pairs of lines,

$$f_1 = 0, \qquad f_2 = 0.\text{*}$$

Hence, any three tangent hyperplanes defined by

$$z = k_i w = 0 \qquad (i = 1, 2, 3) \qquad \ldots \quad (4)$$

will intersect the surface in six line-elements (three pairs) in involution, whose equations are obtained by putting successively k_1, k_2, k_3 for k in (3).

Thus, we obtain the following

Theorem : *The pairs of line-elements through any point on a surface, cut out by the singly infinite number of tangent hyperplanes* (2), *form a system in involution.*

The parameters for the double lines of this involution are obtained by equating the discriminant of (3) to zero, *i.e.*, by

$$H'k^2 - Tk + H = 0. \qquad \ldots \quad (5)$$

Hence, the double lines are obtained by eliminating k between the equations (3) and (5) in the form

$$H'f_1^2 - T\, f_1\, f_2 + H\, f_2^2 = 0 \qquad \ldots \quad (214.1$$

* Salmon, *Conics*, § 342.

or, the double lines are given by

$$\frac{\partial f_1}{\partial x}\cdot\frac{\partial f_2}{\partial y}-\frac{\partial f_1}{\partial y}\cdot\frac{\partial f_2}{\partial x}=0,$$

i.e., $$\vartheta\equiv\begin{vmatrix} ax+hy & hx+by \\ a'x+h'y & h'x+b'y \end{vmatrix}=0. \qquad (214.2$$

This is the equation, which determines the asymptotic directions of the surface (§ 207).

Hence, *the asymptotic directions of the surface are the double lines of the involution of the line-elements through any point of the surface, and the pairs of conjugate elements are harmonic* with these asymptotic directions.*

The hyperplanes which cut out these double lines on the surface may consequently be called the *asymptotic hyperplanes.*

These asymptotic directions are real and distinct, coincident, or imaginary, according as

$$T^2 >, =, \text{ or } < 4HH'.$$

From equations (207.1), it is seen that the Characteristic is a hyperbola, parabola or an ellipse, according as these conditions are satisfied. The point on the surface is accordingly classified into three species, namely, hyperbolic, parabolic or elliptic, and this justifies the use of the word *characteristic* to designate the conic. Hence, the involution has real and distinct, coincident, or imaginary double lines, according as the point is hyperbolic, parabolic or elliptic on the surface.

In the general case of a surface, the asymptotic directions are given by

$$e.dp^2+2f dp dq+g.dq^2=0.$$

* Cremona, *loc. cit.*, § 125.

Hence, these will be real and distinct, coincident, or imaginary, according as

$$f^2 >, =, \text{or}, < eg,$$

and the involution will have double lines, real and distinct, coincident or imaginary (§ 206).

215. The Principal Hyperplanes of Curvature :

Consider the four directions determined by the two tangent hyperplanes

$$f_1 - kf_2 = 0, \qquad f_1 - k'f_2 = 0 \qquad \ldots \quad (1)$$

The condition that these pairs of lines should be harmonic is obtained by equating to zero the mixed invariant * of these, *i.e.*,

$$(a-ka')(b-k'b') + (a-k'a')(b-kb') - 2(h-kh')(h-k'h') = 0$$

or, $kk'(a'b'-h'^2) - (k+k')(ab'+a'b-2hh') + (ab-h^2) = 0$

i.e.,
$$2\mathrm{H}'kk' - (k+k').\mathrm{T} + 2\mathrm{H} = 0. \qquad \ldots \quad (2)$$

The form of this relation shows that the two hyperplanes (1) determine an involution, the double elements being obtained by putting $k=k'$ in (2), *i.e.*, their parameters are given by

$$\mathrm{H}'k^2 - k\mathrm{T} + \mathrm{H} = 0. \qquad \ldots \quad (3)$$

But these are the parameters of the asymptotic hyperplanes, which are, therefore, the *double elements* of the involution determined by (2), and these cut out from the surface two pairs of harmonic line-elements.

By putting $k' = -\frac{1}{k}$ in (2), we obtain one pair of hyperplanes which are mutually orthogonal.† The

* Salmon, *Conics*, § 332.

† Cremona, *loc. cit.*, § 207.

parameters of these hyperplanes are then given by

$$k^2\mathrm{T}-2k(\mathrm{H}-\mathrm{H}')-\mathrm{T}=0.$$

Hence, these hyperplanes are given by the equation

$$\mathrm{T}f^2{}_1-2(\mathrm{H}-\mathrm{H}')f_1f_2-\mathrm{T}f_2{}^2=0. \qquad (215.1$$

These are clearly the bisectors of the angles between the asymptotic hyperplanes (208.2).

Hence, these two hyperplanes may be called the *principal hyperplanes of curvature*.

Thus, *two mutually orthogonal hyperplanes, defined by (215.1), cut out from the surface two pairs of harmonic line-elements and bisect the angle between the asymptotic hyperplanes.*

Note.—At a parabolic point, the two asymptotic hyperplanes coincide with one of the two principal hyperplanes of curvature.

216. Specific Curvature :

In the ordinary space, the specific curvature * of a curved surface at any point has been defined as the average curvature of an infinitely small area including the point. This is the measure of curvature which was shown by Gauss to be the reciprocal of the product of the two principal radii of curvature at the point considered. This is also known as the Gauss's measure of curvature.

If we apply a rotation about the tangent plane $\mathrm{Z}=0$, $\mathrm{W}=0$, through an angle θ, *i.e.*, if we apply the transformation

$$z=z_1\cos\theta+w_1\sin\theta, \qquad w=-z_1\sin\theta+w_1\cos\theta$$

* Also called the *total* curvature, or *Gauss* curvature.

the equations of the approximation-surface now become

$$\left.\begin{aligned} 2z_1 &= (a\cos\theta - a'\sin\theta)x^2 + 2(h\cos\theta - h'\sin\theta)xy \\ &\qquad + (b\cos\theta - b'\sin\theta)y^2 \\ 2w_1 &= (a\sin\theta + a'\cos\theta)x^2 + 2(h\sin\theta + h'\cos\theta)xy \\ &\qquad + (b\sin\theta + b'\cos\theta)y^2 \end{aligned}\right\} \quad \ldots(1)$$

$\therefore$ The equations of the projection-surface in the new tangent hyperplane $z_1 = 0$ are

$$z_1 = 0, \quad 2w_1 = (a\sin\theta + a'\cos\theta)x^2 + 2(h\sin\theta + h'\cos\theta)xy + (b\sin\theta + b'\cos\theta)y^2.$$

If, then, ρ_1, ρ_2 denote the principal radii of curvature of this surface,* we obtain

$$\frac{1}{\rho_1\rho_2} = (a\sin\theta + a'\cos\theta)(b\sin\theta + b'\cos\theta) - (h\sin\theta + h'\cos\theta)^2$$

$$i.e.,\ \frac{1}{\rho_1\rho_2} = \mathrm{H}\sin^2\theta + \mathrm{T}\sin\theta.\cos\theta + \mathrm{H}'\cos^2\theta. \quad (216.1$$

The form of this expression shows that $\dfrac{1}{\rho_1\rho_2}$ is equal to the reciprocal of the radius of curvature of the normal section of the surface

$$2z = \mathrm{H}'x^2 + \mathrm{T}xy + \mathrm{H}y^2 \quad \ldots \quad (2)$$

in the hyperplane $w = 0$ made by a plane through the axis of z, and making an angle θ with the axis of x.

Hence, we conclude that the expression $\dfrac{1}{\rho_1\rho_2}$ may be treated in the same manner as the radius of curvature for a normal section of a three-dimensional surface. Therefore, corresponding to the two principal radii of

* This may be deduced from § 295, Salmon's *Geometry of Three Dimensions*, by putting $\mathrm{L} = \mathrm{M} = 0$, $\mathrm{N} = 1$ in that formula.

curvature of the ordinary surface in $w=0$, there are two maximum and minimum values of $\dfrac{1}{\rho_1\rho_2}$ for all the projection-surfaces in the tangent hyperplanes with other specific curvatures zero.

The tangent hyperplane giving the maximum and minimum specific curvatures is obtained from (216.1), by putting

$$\frac{\partial}{\partial\theta}\left(\frac{1}{\rho_1\rho_2}\right)=0$$

which gives

$$2\mathrm{H}\ \sin\theta\cos\theta+\mathrm{T}\,(\cos^2\theta-\sin^2\theta)-2\mathrm{H}'\cos\theta\sin\theta=0$$

$$\text{or,}\qquad \mathrm{T}\cos 2\theta=(\mathrm{H}'-\mathrm{H})\sin 2\theta$$

$$i.e.,\qquad \tan 2\theta=\frac{\mathrm{T}}{\mathrm{H}'-\mathrm{H}}. \qquad \ldots \quad (3)$$

This gives two values of $\tan\theta$ whose product is equal to -1. Hence, the two hyperplanes, which give the maximum and minimum values of the specific curvature, or what is the samething, which give the two principal specific curvatures, are mutually orthogonal.

From what has been said previously (§ 215), it will be noticed that these are the two hyperplanes of principal curvature.

By putting $\dfrac{1}{\rho_1\rho_2}=0$, we obtain from (216.1)

$$\mathrm{H}\sin^2\theta+\mathrm{T}\sin\theta\cos\theta+\mathrm{H}'\cos^2\theta=0$$

which gives $$\tan\theta=-\frac{\mathrm{T}\pm\sqrt{\mathrm{T}^2-4\mathrm{HH}'}}{2\mathrm{H}}. \qquad \ldots \quad (4)$$

Hence, these values of θ give two tangent hyperplanes, which correspond to zero specific curvature, and they may be called *Hyperplanes of zero curvature*, or,

Null Hyperplanes. These two are found to be orthogonal to the two asymptotic hyperplanes (§ 214). Since, as has been shown, the hyperplanes of principal curvature bisect the angle between the asymptotic hyperplanes, they also bisect the angle between these two hyperplanes of zero specific curvature.

By putting $\theta+\frac{1}{2}\pi$ in place of θ, we obtain the specific curvature $\frac{1}{\rho_1'\rho_2'}$ of the projection-surface in $w_1=0$ in the form

$$\frac{1}{\rho_1'\rho_2'}=\mathrm{H}\cos^2\theta-\mathrm{T}\sin\theta\cos\theta+\mathrm{H}'\sin^2\theta$$

$$\therefore \qquad \frac{1}{\rho_1\rho_2}+\frac{1}{\rho_1'\rho_2'}=\mathrm{H}+\mathrm{H}' \qquad (216.2$$

i. e., The sum of the specific curvatures of two projection-surfaces in two mutually orthogonal tangent hyperplanes is constant.

Since Gauss curvature can be determined from the line-element $ds^2=\mathrm{E}dp^2+2\mathrm{F}dpdq+\mathrm{G}dq^2$, this sum for a general surface can also be obtained from the same formula. Consequently, the above sum remains invariant for any deformation of the surface. This may be called the *mean curvature* of the surface in the fourfold.

217. Geodesic Lines:

Geodesic lines are among the most interesting organic curves, which can be drawn on a surface, and have long engaged the attention of mathematicians. The principal properties of these curves are intrinsically

* See Forsyth, *Lectures on the Differential Geometry of Curves and Surfaces*, § 133; also *Geometry of Four Dimensions*, § 319.

associated with, and easily derivable from, the curvature properties of surfaces. The formulae used in the three-dimensional geometry require to be modified here, so as to be applicable to the geometry of the fourfold.

A geodesic line on a surface is variously defined, but all these definitions lead to one fundamental property that it is a curve such that its arc between any two neighbouring points is shorter than the arc of any other curve on the surface joining the same two points, or what is the same thing, it is the curve of shortest distance, measured in the surface between two points.*

Properties of geodesics are most conveniently studied in differential geometry by applying principles of differential equations. There is, therefore, no scope for their detailed discussion in the present volume, but an introduction of the preliminary notions seems desirable, with a view to indicate the possibility of extension of the properties in higher dimensions.

It can be inferred from the definition that a geodesic on a surface is a curve whose osculating plane at any point contains the normal to the surface at that point. Since there is no unique normal at any point of a surface in the fourfold, this definition is not helpful in establishing its properties. For analytical discussion, therefore, the former definition is most suitable, and we accordingly define a geodesic curve as *the line of minimum length between two points on a surface measured along the surface*. One mode of treatment is afforded by the fact that for an indefinitely small arc, the chord of which is given, the excess in length

* In the ordinary space, the property is analytically established by making the curve satisfy different critical tests due to Legendre, Jacobi and Weierstrass.

over the chord is so much the less as the radius of curvature is greater. The shortest arc, therefore, joining two indefinitely near points A, B on a surface is that which has the greatest radius of curvature, and this is evidently the normal section. Thus, the fundamental property of geodesics follows from Meusnier's Theorem.*

218. Differential Equations of Geodesics:

An element of arc is given by the expression

$$ds^2 = \mathrm{E}dp^2 + 2\mathrm{F}dp\,dq + 2\mathrm{G}dq^2.$$

Since the *arc* is to be made "geodesic," it is desirable to introduce a new variable t, instead of taking s as an independent variable. We write

$$p_1 = \frac{dp}{dt}, \quad q_1 = \frac{dq}{dt}$$

$$\therefore \qquad \frac{ds^2}{dt^2} = \mathrm{E}\left(\frac{dp}{dt}\right)^2 + 2\mathrm{F}\left(\frac{dp}{dt}\cdot\frac{dq}{dt}\right) + \mathrm{G}\left(\frac{dq}{dt}\right)$$

$$= \mathrm{E}p_1^2 + 2\mathrm{F}p_1q_1 + \mathrm{G}q_1^2$$

whence $$ds = (\mathrm{E}p_1^2 + 2\mathrm{F}p_1q_1 + \mathrm{G}q_1^2)^{\frac{1}{2}}.dt.$$

$$\therefore \quad s = \int (\mathrm{E}p_1^2 + 2\mathrm{F}p_1q_1 + \mathrm{G}q_1^2)^{\frac{1}{2}} dt \equiv \int f dt \text{ (say)}. \quad \dots \quad (1)$$

For a geodesic, this integral has to be made a minimum, where E, F, G are known functions of p and q, assuming that the curve is restricted to a portion of the surface, where there is a one-to-one correspondence between the surface and its image in the pq plane, and that the portion of the surface is free from singular points, *i.e.*, $\mathrm{EG} - \mathrm{F}^2 > 0$. Using Weierstrass's form of

* Bertrand, *Liouville*, t. XIII, p. 73, cited by Cayley, *Quarterly Journal*, Vol. 1, p. 186.

Euler's equations,* we obtain

$$f_p - \frac{d}{dt}(f_{p_1}) = 0, \qquad f_q - \frac{d}{dt}(f_{q_1}) = 0. \quad \ldots \ (2)$$

These two are not independent, but are equivalent to one differential equation.

Having thus formed the characteristic equations, the arc may now be taken for the parameter t, and the extremals are obtained from the two differential equations

$$\left.\begin{aligned} 2\frac{d}{ds}(Ep' + Fq') &= E_p p'^2 + 2F_p p'q' + G_p q'^2 \\ 2\frac{d}{ds}(Fp' + Gp') &= E_q p'^2 + 2F_q p'q' + G_q q'^2 \end{aligned}\right\} \qquad (218.1$$

These are the equations of a *geodesic curve* upon the surface. The equation of a geodesic may be derived in another form, if it be defined by the geometrical property that its geodesic curvature is constantly zero.

The geodesic curvature of the curve at the point t

$$p = \phi(t), \qquad q = \psi(t)$$

is found to be given by the expression †

$$\frac{1}{\rho} = \frac{\vartheta}{\sqrt{EG - F^2}\,(\sqrt{Ep'^2 + 2Fp'q' + Gp'^2})^3}$$

where $\vartheta \equiv (EG - F^2)(p'q'' - q'p'')$

$$+ (Ep' + Fq')[(F_p - \tfrac{1}{2}E_q)p'^2 + G_p p'q' + \tfrac{1}{2}G_q q'^2]$$
$$- (Fp' + Gq')[\tfrac{1}{2}E_p p'^2 + E_q p'q' + (F_q - \tfrac{1}{2}G_q)q'^2]$$

Hence, the extremals satisfy the differential equation

$$\vartheta = 0. \qquad \ldots \ (218.2$$

Also, putting $\phi(f)$ for the combined form of Euler's equations, we obtain—

$$\phi(\sqrt{Ep'^2 + 2Fp'q' + Gq'^2}) = \frac{\vartheta}{(\sqrt{Ep'^2 + 2Fp'q' + Gq'^2})^3} = 0 \ \ldots (3)$$

* Kneser, *Lehrbuch der Variationsrechuung* (1900).

† Oskar Bolza, *Concerning the Isoperimetric Problem on a given Surface*, Decennial Publications of the University of Chicago, Vol. IX, p. 13.

The differential equations (218.1) may be combined into another form as follows:

We make use of the following abbreviations:

$$V^2\Gamma = mG - nF, \qquad V^2\Delta = nE - mF$$

$$V^2\Gamma' = m'G - n'F, \qquad V^2\Delta' = n'E - m'F$$

$$V^2\Gamma'' = m''G - n''F, \qquad V^2\Delta'' = n''E - m''F$$

$$m = \Sigma x_1 x_{11}, \quad m' = \Sigma x_1 x_{12}, \quad m'' = \Sigma x_1 x_{22}$$

$$n = \Sigma x_2 x_{11}, \quad n' = \Sigma x_2 x_{12}, \quad n'' = \Sigma x_2 x_{22}$$

The two equivalent forms of equations (2), derivable from Euler's equations, may ultimately be reduced to the forms *

$$\left.\begin{aligned} p'' + \Gamma p'^2 + 2\Gamma' p'q' + \Gamma'' q'^2 = 0 \\ q'' + \Delta p'^2 + 2\Delta' p'q' + \Delta'' q'^2 = 0 \end{aligned}\right\} \qquad (218.3$$

These are the equations of a geodesic curve upon a surface. If p is taken as the independent variable, the above two equivalent forms may be combined into one, namely,

$$\frac{d^2q}{dp^2} = \Gamma''\left(\frac{dq}{dp}\right)^3 + (2\Gamma' - \Delta'')\left(\frac{dq}{dp}\right)^2 + (\Gamma - 2\Delta')\frac{dq}{dp} - \Delta \qquad (218.4$$

Note.—The theory of differential equations shows that the primitive of the equations, expressing p and q as functions of t, involves four arbitrary constants, three of which are determinable uniquely by given values of p, q and $\frac{dp}{dq}$, corresponding to an assigned value of t. Hence, a geodesic is uniquely determined by any point and its direction at that point.

* Forsyth, *Geometry of Four Dimensions*, Vol. 1, § 219, p. 381.

219. Lines of Curvature:

In connection with the curvature of surfaces, the most important lines which can be traced on a surface are the *lines of curvature*. A line of curvature, as defined in the ordinary space, is a curve traced on a surface such that the tangent to the curve at any point is also a tangent to one of the principal normal sections of the surface at that point. Hence, since there are two mutually orthogonal normal sections, there are two lines of curvature through any point mutually orthogonal. This definition is easily shown to be equivalent to this, that a line of curvature is one such that the normals to the surface at any two consecutive points of the curve intersect each other.

In the fourfold, there is no unique normal direction of the surface at any point, but there is instead a normal plane, and the definition can be so extended that the normal planes at two consecutive points on a line of curvature will intersect, as in fact they do; but this is no distinguishing property, for all curves through any point possess this property, as has already been shown.

If, however, they proceed along the four principal directions of curvature from point to point, we obtain a quadruply infinite system of curves on the surface, which may be defined as the *lines of curvature* at any point.

In order to obtain the differential equation of the lines of curvature, we note that the point of intersection of consecutive normal planes in the direction of a particular principal section lies in the osculating plane of each normal section through this direction.

Let the equations of a skew curve be denoted by

$$x=x(p), \quad y=y(p), \quad z=z(p), \quad w=w(p),$$

where p is a parameter. Then the equations of the osculating plane at the point (p) are given by *

$$X-x=\lambda\frac{dx}{dp}+\mu\frac{d^2x}{dp^2}, \quad Y-y=\lambda\frac{dy}{dp}+\mu\frac{d^2y}{dp^2}, \text{ etc.}$$

where X, Y, Z, W are the current co-ordinates, and λ, μ are variable parameters. Now, if we consider q as a function of p, we shall obtain the equations of the osculating plane of a curve passing through the point (p, q) on the surface in the forms

$$X-x=\lambda x_1+\mu\left(\frac{dx_1}{dp}+\frac{dx_2}{dp}\cdot\frac{dq}{dp}\right)+x_2\left(\lambda\frac{dq}{dp}+\mu\frac{d^2q}{dp^2}\right) \quad \ldots \quad (1)$$

with similar expressions for $Y-y$, $Z-z$ and $W-w$.

Eliminating λ, μ, $\frac{dq}{dp}$, $\frac{d^2q}{dp^2}$ from equations (1), we obtain the determinant equation

$$\begin{vmatrix} X-x & x_1 & x_2 & dx_1dp+dx_2dq \\ Y-y & y_1 & y_2 & dy_1dp+dy_2dq \\ Z-z & z_1 & z_2 & dz_1dp+dz_2dq \\ W-w & w_1 & w_2 & dw_1dp+dw_2dq \end{vmatrix}=0. \quad \ldots(2)$$

This shows that the osculating plane (1) is contained in the hyperplane (2). But this equation is identically satisfied by the co-ordinates of any point in the tangent plane (§ 198), and consequently, it is a tangent hyperplane at the point (p, q), and we may regard the equation (2) and the equation of the normal hyperplane,

* See Chap. X, § 225.

corresponding to the direction $\frac{dq}{dp}$, as defining the osculating plane of the normal section. The point (X, Y, Z, W) of intersection of consecutive normal planes in the assumed direction $\frac{dq}{dp}$ will, therefore, satisfy both these equations. Since the point (X, Y, Z, W) lies in the normal plane of the point (p, q), its co-ordinates will satisfy the equations of the normal plane of the same. Hence, they will satisfy equation (2).

By using fundamental magnitudes of the second order, we may write equation (2) in the form

$$\Sigma(X-x)(\vartheta_x dp^2 + 2\Gamma_x dpdq + \Lambda_x dq^2) = 0. \qquad \dots \quad (3)$$

From the parametric equation (§ 206) of the Characteristic and this relation (3), we obtain the differential equation of the lines of curvature in the form

$$\Sigma\nabla_x\{(E\Gamma_x - F\vartheta_x)dp^2 + (E\Lambda_x - G\vartheta_x)dpdq + (F\Lambda_x - G\Gamma_x)dq^2\} = 0 \quad (219.1$$

where $\nabla_x \equiv (\vartheta_x dp^2 + 2\Gamma_x dpdq + \Lambda_x dq^2)$.

The equations (1) suggest another method for finding the radius of principal curvature as follows :—

$$\rho^2 = (X-x)^2 + (Y-y)^2 + (Z-z)^2 + (W-w)^2,$$

where ρ is the radius of curvature, corresponding to the direction (dp, dq).

Squaring the determinant (2), taking columns with columns, we get, in virtue of the relations of § 206,

$$\begin{vmatrix} \rho^2 & 0 & 0 & ds^2 \\ 0 & E & F & g'' \\ 0 & F & G & h'' \\ ds^2 & g'' & h'' & \Sigma a''^2 \end{vmatrix} = 0 \qquad \dots \quad (4)$$

where $a'' \equiv dx_1 dp + dx_2 dq \qquad c'' \equiv dz_1 dp + dz_2 dq$

$$b'' \equiv dy_1 dp + dy_2 dq \qquad d'' \equiv dw_1 dp + dw_2 dq$$

$$g'' \equiv dp\Sigma x_1 dx_1 + dq\Sigma x_1 dx_2$$

$$h'' \equiv dp\Sigma x_2 dx_1 + dq\Sigma x_2 dx_2.$$

But
$$\left\| \begin{matrix} x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ a'' & b'' & c'' & d'' \end{matrix} \right\|^2 = \begin{vmatrix} \mathrm{E} & \mathrm{F} & g'' \\ \mathrm{F} & \mathrm{G} & h'' \\ g'' & h'' & \Sigma a''^2 \end{vmatrix}$$

$$= \Sigma \begin{vmatrix} y_1 & z_1 & w_1 \\ y_2 & z_2 & w_2 \\ b'' & c'' & d'' \end{vmatrix}^2$$

$$= \Sigma(\vartheta_x dp^2 + 2\Gamma_x dpdq + \wedge_x dq^2)^2 = \Sigma\nabla_x.$$

$$\therefore \quad \rho^2 = \frac{(\mathrm{E}dp^2 + 2\mathrm{F}dpdq + \mathrm{G}dq^2)^2(\mathrm{EG} - \mathrm{F}^2)}{\Sigma(\vartheta_x dp^2 + 2\Gamma_x dpdq + \wedge_x dq^2)^2}. \qquad (219.2$$

This gives four principal radii of curvature, corresponding to the four roots of (219.1), at any point (p, q) of the surface.

220. Curvature Properties of a Surface:

From the foregoing analysis we are now in a position to summarise the following facts with regard to the mutual relations between the loci associated with a surface-point and the curvature properties of the surface at that point:

(1) Through every direction through a point O on a surface, there is a *geodesic*, whose centre of curvature

is a point Q. This is called the *geodesic centre* and lies in the normal plane.

(2) Corresponding to every direction through O, there is a point P in the normal plane, which is its intersection with the corresponding consecutive normal plane, and may be called the *orthogonal* centre.

(3) The triangle OQP is right-angled at Q, and the line QP is parallel to the normal to the osculating hyperplane corresponding to the given direction.

(4) The locus of Q is a bicircular quartic, and is the locus of the geodesic centre. It touches the locus of P, which is a conic (*i.e.*, the Characteristic) at four points and these are the four centres of principal curvature of the surface. The locus of the geodesic centre is the pedal of the Characteristic with respect to O.

(5) There are four directions through the point which are *lines of curvature*, and give the maximum and minimum values of the four radii of principal curvature. These values are given in direction and magnitude by the four radii, drawn from O to the four points of contact on the Characteristic. These four directions are normal to the Characteristic, and the lengths of these four normals are the four radii of curvature of the corresponding four principal geodesics.

(6) There is another conic, which is the inverse of the pedal, *i.e.*, the reciprocal polar of the Characteristic and is called the *Indicatrix*. It is the locus of points, whose radii represent measures of curvature of the geodesics and not the radii of curvature.

CHAPTER X

Curves in the Fourfold

221. Skew Curves:

Of all curved loci in the fourfold, properties of curves have been exhaustively studied by several workers * from different standpoints, specially with reference to their different curvatures. The method of treatment in almost every case is differential, which appears to be the only fruitful method for discussing intrinsic properties of geometrical loci. We shall conclude the present work with a brief discussion of the fundamental properties of a curve in the fourfold, for the most part as extension of known properties usually discussed in the differential geometry of three dimensions.

A line or curve may lie entirely in a plane, or may be completely contained in a hyperplane, or in the most general case, it may lie in an infinite number of distinct hyperplanes, all contained in the general fourfold. The nature of these three distinct categories of curves is appropriately expressed by endowing them with characteristic properties, which may be described under the general name of possessing *curvatures*. In the case of a plane curve, it is said to possess *one* curvature, in the case of a space-curve (a curve lying in a hyperplane), it is described as possessing a *second* curvature, namely, *torsion*. Finally, in the general case, a curve is said

* See the paper by G. Pirondini, *Giorn. mat. Napoli*, Vol. 28, pp. 219-239.

to possess an additional *third* curvature, which may be called "*tilt*." * Thus, a plane curve has one *curvature*, a space-curve has double curvature—*curvature and torsion*, and a general curve in the fourfold has triple curvature, namely, *curvature*, *torsion* and *tilt*.

222. Representation of a Curve:

There are various modes of analytically representing a curve. Just as two equations—not both linear—in three variables are taken to represent a skew-curve in the ordinary space, so in the fourfold we may take three independent equations—not all linear—of the forms

$$f_1(x, y, z, w)=0, \quad f_2(x, y, z, w)=0, \quad f_3(x, y, z, w)=0$$

to represent a curve. These equations, however, may represent two or more curves, which are geometrically discrete, although analytically represented by the same equations.

By imposing one restriction upon the movement of a point in the fourfold, by means of one relation between the co-ordinates, the co-ordinates of such a point are capable of being represented in terms of three parameters in the forms

$$x=x(p, q, r), \quad y=y(p, q, r), \quad z=z(p, q, r), \quad w=x(p, q, r).$$

In this case, the points (and consequently the curve) are restricted to lie in a hypersurface, and the curve is determined by two relations of the forms

$$\phi(p, q, r)=0, \quad \psi(p, q, r)=0.$$

* The nomenclature is due to Prof. Forsyth.

If, however, another restriction is imposed, the co-ordinates of a point are expressible in terms of two parameters, and the points lie on a surface. The curve is then determined by means of only one relation between the two parameters in the form

$$\chi\,(p,\,q)=0.$$

In each of the above representations, it will be noticed, that the variables are ultimately reducible to functions of a single variable parameter whatever its geometrical significance. In the study of the intrinsic properties of a curve, an intrinsic variable related to the curve may conveniently be taken as the variable parameter ; for instance, the length of the arc of the curve, measured along the curve from some fixed point of reference, may be taken as the independent variable, and the co-ordinates of any point on the curve may be expressed as functions of this parametric variable (arc) in the forms

$$x=f_1(s),\quad y=f_2(s),\quad z=f_3(s),\quad w=f_4(s) \qquad (222.1$$

where s is the length of the arc, measured along the curve from a fixed point O.

223. Associated Lines, Planes and Hyperplanes :

Two consecutive points on a curve determine a line, which is called the *tangent line* at the point. Three consecutive points determine a plane, which is called the *osculating plane* at the point. Four consecutive points on the curve determine a hyperplane, which may be called the *osculating hyperplane* at the point.

At each point of a curve there is a ∞^2 of lines perpendicular to the tangent line. These lines are called

'*normals*' to the curve at the point, and they determine a hyperplane, which may be called the *normal hyperplane* of the curve at the point. Of these normals, there is one, and only one, which lies in the osculating plane, and is called the *principal normal*.

There is a ∞^1 of lines, which are all orthogonal to the osculating plane and are called *binormals*. They determine a plane, absolutely orthogonal to the osculating plane. There is one, and only one, of these binormals, which lies in the osculating hyperplane and is called the *principal binormal*. The plane of the binormals may be called the *binormal plane*.

There is only one line orthogonal to the osculating hyperplane determined by three consecutive tangent lines to the curve and is called the *trinormal*. The normal hyperplane and the osculating hyperplane at any point of a curve have a common plane of section, which may be called the *principal normal plane*. In fact, there is a ∞^1 of normal planes, all situated in the normal hyperplane, and there is one, and only one, of these planes which lies in the osculating hyperplane and is called the *principal normal plane*.

The principal normal and the principal binormal lie in the normal hyperplane and the osculating hyperplane. The tangent line, the principal normal, the principal binormal and the trinormal at a given point of a curve are the *principal directions*, and form an orthogonal frame associated with any curve. It will be noticed that the osculating hyperplane contains the tangent line, the principal normal and the principal binormal, while the trinormal is perpendicular to the osculating hyperplane.

224. The Tangent Line :

Let **P** be a point on the curve with co-ordinates (x, y, z, w). Then the co-ordinates of a contiguous point $\mathbf{P}'$ may be taken as $(x+dx, y+dy, z+dz, w+dw)$, where the variations of the co-ordinates of $\mathbf{P}'$ from those of **P** are quantities of the first order of smallness.

The length of the arc ds from **P** to $\mathbf{P}'$ is given by

$$\begin{aligned} ds^2 &= \{(x+dx)-x\}^2+\{(y+dy)-y\}^2 \\ &\qquad +\{(z+dz)-z\}^2+\{(w+dw)-w\}^2 \\ &= dx^2+dy^2+dz^2+dw^2. \end{aligned} \tag{224.1}$$

If x', y', z', w' respectively stand for $\dfrac{dx}{ds}, \dfrac{dy}{ds}, \dfrac{dz}{ds}, \dfrac{dw}{ds}$,

then, we get
$$x'^2+y'^2+z'^2+w'^2=1. \tag{224.2}$$

Denoting the current co-ordinates by X, Y, Z, W, the equations of the tangent line at **P** (x, y, z, w) may be written as

$$\frac{X-x}{(x+dx)-x}=\frac{Y-y}{(y+dy)-y}=\frac{Z-z}{(z+dz)-z}=\frac{W-w}{(w+dw)-w}$$

$$\text{or,}\quad \frac{X-x}{x'}=\frac{Y-y}{y'}=\frac{Z-z}{z'}=\frac{W-w}{w'}. \tag{224.3}$$

Since $\Sigma x'^2=1$, (x', y', z', w') are certainly the direction-cosines of the tangent line. The equations of the normal hyperplane is, then,

$$(X-x)x'+(Y-y)y'+(Z-z)z'+(W-w)w'=0. \tag{224.4}$$

225. The osculating Plane :

The co-ordinates of a third point $\mathbf{P}''$ contiguous to $\mathbf{P}'$ may be taken as—

$$\begin{array}{ll} x+dx+d(x+dx), & y+dy+d(y+dy), \\ z+dz+d(z+dz), & w+dw+d\,(w+dw) \end{array}$$

i.e., $(x+2dx+d^2x, y+2dy+d^2y, z+2dz+d^2z, w+2dw+d^2w)$

so that the direction-cosines of the line $\mathbf{P}'$ $\mathbf{P}''$ are proportional to

$$dx+d^2x, \quad dy+d^2y, \quad dz+d^2z, \quad dw+d^2w$$

i. e., proportional to

$$x'+x''ds, \quad y'+y''ds, \quad z'+z''ds, \quad w'+w''ds$$

to the first order of small quantities,

where $\quad x''=\dfrac{d^2x}{ds^2}, \quad y''=\dfrac{d^2y}{ds^2}, \quad z''=\dfrac{d^2z}{ds^2}, \quad w''=\dfrac{d^2w}{ds^2}.$

Hence, the equations of the plane through the three points $\mathbf{P}$, $\mathbf{P}'$, $\mathbf{P}''$, and consequently, containing the tangent line at $\mathbf{P}$ and the consecutive tangent line at the contiguous point $\mathbf{P}'$ are obtained in the form

$$\left\|\begin{matrix} X-x & Y-y & Z-z & W-w \\ x' & y' & z' & w' \\ x'' & y'' & z'' & w'' \end{matrix}\right\| = 0. \qquad (225.1$$

Thus, the osculating plane passes through three consecutive points, or contains two consecutive tangent lines to the curve. As the plane contains two consecutive tangents, sufficient to determine a plane through $\mathbf{P}$, it has the closest possible contact for any plane at $\mathbf{P}$. It is, therefore, the *osculating plane*.

226. The osculating Hyperplane :

By an extension of the ordinary formulæ, the co-ordinates of any point contiguous to $\mathbf{P}$ may be taken as

$$X=x+x'ds+\tfrac{1}{2}x''ds^2+\tfrac{1}{6}x'''ds^3+\ldots\ldots$$
$$Y=y+y'ds+\tfrac{1}{2}y''ds^2+\tfrac{1}{6}y'''ds^3+\ldots\ldots$$
$$Z=z+z'ds+\tfrac{1}{2}z''ds^2+\tfrac{1}{6}z'''ds^3+\ldots\ldots$$
$$W=w+w'ds+\tfrac{1}{2}w''ds^2+\tfrac{1}{6}w'''ds^3+\ldots\ldots$$

where s denotes the arc **PP**$'$ measured along the curve.

The equation of a hyperplane through **P**(x, y, z, w) may be written as—

$$L(X-x)+M(Y-y)+N(Z-z)+P(W-w)=0 \quad \ldots \quad (1)$$

If this passes through three points consecutive to **P** the equation (1) will be satisfied by these values of the co-ordinates, when the co-efficients of the first three powers of ds, namely, ds, ds^2, ds^3, are made to vanish, and higher powers do not vanish. Thus, equating the co-efficients of ds, ds^2, ds^3 to zero, we obtain

$$Lx'+My'+Nz'+Pw'=0$$

$$Lx''+My''+Nz''+Pw''=0$$

$$Lx'''+My'''+Nz'''+Pw'''=0$$

but $$Lx^{iv}+My^{iv}+Nz^{iv}+Pw^{iv}\neq 0$$

Eliminating L, M, N, P between these relations and (1), the equation of the osculating hyperplane is obtained in the form

$$\begin{vmatrix} X-x & Y-y & Z-z & W-w \\ x' & y' & z' & w' \\ x'' & y'' & z'' & w'' \\ x''' & y''' & z''' & w''' \end{vmatrix}=0. \qquad (226.1$$

This equation may also be obtained from the fact that the hyperplane in question contains three consecutive tangent lines, and it is then uniquely determined. Hence, this hyperplane has the greatest possible contact with the curve than any other hyperplane at the point **P**, and is, therefore, called the *osculating hyperplane*. If $T_x, -T_y, T_z, -T_w$ denote the

direction-cosines of the trinormal (normal to the osculating hyperplane) T_x, T_y, T_z, T_w are the four determinants of the matrix

$$\left\| \begin{matrix} x' & y' & z' & w' \\ x'' & y'' & z'' & w'' \\ x''' & y''' & z''' & w''' \end{matrix} \right\| \qquad (226.2$$

227. The Binormal Plane:

The osculating plane contains two lines, whose direction-cosines are proportional to x', y', z', w' and x'', y'', z'', w''. The equations of two hyperplanes through **P** at right angles to these directions are

$$\left.\begin{matrix} (X-x)x'+(Y-y)y'+(Z-z)z'+(W-w)w'=0 \\ (X-x)x''+(Y-y)y''+(Z-z)z''+(W-w)w''=0 \end{matrix}\right\} \quad \ldots \quad (1)$$

These two hyperplanes determine, by their intersection, a plane, which is absolutely orthogonal to the osculating plane. For, any direction $(\lambda, \mu, \nu, \rho)$ lying in this plane satisfies the relations

$$\Sigma\lambda x'=0, \qquad \Sigma\lambda x''=0,$$

showing that the direction $(\lambda, \mu, \nu, \rho)$ is perpendicular to two directions in the osculating plane and is, therefore, normal to the latter.

Hence, the plane determined by the equations (1) is the *binormal plane*, and is absolutely perpendicular to the osculating plane.

It can be easily shown that this plane is parallel to the plane in which the normal hyperplanes at two consecutive points intersect. For, the normal hyperplane at **P** is given by

$$\Sigma(X-x)x'=0 \qquad \ldots \quad (2)$$

and the normal hyperplane at the consecutive point $\mathbf{P}'$ is given by

$$\Sigma(X - x - x'ds)(x + x''ds) = 0$$

$$i.e.,\ \Sigma(X - x)x' + [\{\Sigma(X - x)x''\} - 1]ds = 0. \qquad \dots \quad (3)$$

The equations of the plane of intersection of these two hyperplanes (2) and (3) in the limit are

$$\Sigma(X - x)x' = 0, \qquad \Sigma(X - x)x'' = 1 \qquad \dots \quad (4)$$

showing that this plane is *parallel* to the binormal plane.

228. The Principal Normal :

The osculating plane at any point $\mathbf{P}$ of a curve intersects the normal hyperplane in a line, which is called the *Principal Normal*.

Any direction in the osculating plane may be taken as

$$\alpha x' + \beta x'', \quad \alpha y' + \beta y'', \quad \alpha z' + \beta z'', \quad \alpha w' + \beta w''.$$

If this lies in the normal hyperplane (2), we must have

$$(\alpha x' + \beta x'')x' + (\alpha y' + \beta y'')y' + (\alpha z' + \beta z'')z' + (\alpha w' + \beta w'')w' = 0$$

$$\text{or,}\quad \alpha(x'^2 + y'^2 + z'^2 + w'^2) + \beta(x'x'' + y'y'' + z'z'' + w'w'') = 0.$$

Since $\Sigma x'^2 = 1$, this relation will be satisfied, if

$$\alpha = 0, \qquad \Sigma x'x'' = 0.$$

Hence, the direction-cosines of the line of intersection, *i.e.*, the principal normal are

$$\beta x'', \ \beta y'', \ \beta z'', \ \beta w'', \quad \text{where} \quad \Sigma(\beta x'')^2 = 1,$$

$$\text{or,}\ \Sigma x''^2 = x''^2 + y''^2 + z''^2 + w''^2 = \frac{1}{\beta^2} = \frac{1}{\rho^2}\ (\text{say}). \quad \dots \quad (1)$$

The significance of using ρ for the constant will appear in the sequel.

Thus, the direction-cosines of the prinicpal normal are found to be proportional to x'', y'', z'', w'', and its equations can, therefore, be written as

$$\frac{X-x}{x''}=\frac{Y-y}{y''}=\frac{Z-z}{z''}=\frac{W-w}{w''} \qquad (228.1$$

the direction-cosines being $\rho x''$, $\rho y''$, $\rho z''$, $\rho w''$.

229. The Principal Binormal:

The binormal plane intersects the osculating hyperplane in a line, which is called the *Principal Binormal* of the curve at **P**.

Any direction L, M, N, P being taken through **P** in the osculating hyperplane, we may write

$$\left.\begin{aligned} L&=\alpha x'+\beta x''+\gamma x''' & N&=\alpha z'+\beta z''+\gamma z''' \\ M&=\alpha y'+\beta y''+\gamma y''' & P&=\alpha w'+\beta w''+\gamma w''' \end{aligned}\right\} \quad \ldots \quad (1)$$

where $\Sigma L^2=\alpha^2\Sigma x'^2+\beta^2\Sigma x''^2+\gamma^2\Sigma x'''^2$

$$+2\alpha\beta\Sigma x'x''+2\alpha\gamma\Sigma x'x'''+2\beta\gamma\Sigma x''x'''$$

$$=1. \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

If this line also lies in the binormal plane, it is at right angles to the two normals of the generating hyperplanes of that plane, and we must have

$$Lx'+My'+Nz'+Pw'=0 \quad \ldots \quad (3)$$

$$Lx''+My''+Nz''+Pw''=0. \quad \ldots \quad (4)$$

From (1), we obtain the two following relations:—

$$\alpha\Sigma x'^2+\beta\Sigma x'x''+\gamma\Sigma x'x'''=0$$

$$\alpha\Sigma x'x''+\beta\Sigma x''^2+\gamma\Sigma x''x'''=0.$$

Since $\qquad x''^2+y''^2+z''^2+w''^2=\frac{1}{\rho^2}$, we obtain

$$x''x'''+y''y'''+z''z'''+w''w'''=\frac{-\rho'}{\rho^3}. \quad \ldots \quad (5)$$

Also, $\Sigma x'x''=0$, whence $\Sigma x'x'''+\Sigma x''^2=0$, or, $\Sigma x'x'''=-\dfrac{1}{\rho^2}$.

$$\therefore \qquad \alpha-\frac{\gamma}{\rho^2}=0, \text{ and } \frac{\beta}{\rho^2}-\frac{\gamma\rho'}{\rho^3}=0,$$

$$i.e., \qquad \alpha\rho^2=\gamma, \qquad \beta\rho=\gamma\rho'.$$

Substituting these values in (2), we obtain

$$\Sigma L^2=1=\alpha^2+\frac{\beta^2}{\rho^2}+\gamma^2\Sigma x'''^2-2\alpha\gamma\,\frac{1}{\rho^2}-2\beta\gamma\cdot\frac{\rho'}{\rho^3}$$

$$=\frac{\gamma^2}{\rho^4}+\frac{\gamma^2}{\rho^2}\frac{\rho'^2}{\rho^2}+\gamma^2\Sigma x'''^2-2\frac{\gamma^2}{\rho^2}\cdot\frac{1}{\rho^2}-2.\gamma^2\cdot\frac{\rho'^2}{\rho^4}$$

$$=\frac{\gamma^2}{\rho^4}\{1+\rho'^2+\rho^4\Sigma x'''^2-2-2\rho'^2\}$$

$$=\frac{\gamma^2}{\rho^4}\{\rho^4\Sigma x'''^2-1-\rho'^2\}$$

$$=\frac{\gamma^2}{\rho^2}\left\{\rho^2\Sigma x'''^2-\frac{1}{\rho^2}-\frac{\rho'^2}{\rho^2}\right\}.$$

If, now, we put $1/\sigma^2=\rho^2\Sigma x'''^2-\dfrac{1}{\rho^2}-\dfrac{\rho'^2}{\rho^2}$, we obtain

$$1=\frac{\gamma^2}{\rho^2\sigma^2}, \text{ whence } \gamma^2=\rho^2\sigma^2, \quad \alpha=\sigma/\rho, \quad \beta=\sigma\rho'.$$

Making these substitutions in (1), the direction common between the osculating plane and the osculating hyperplane may be taken as

$$\left.\begin{array}{ll} L=\dfrac{\sigma}{\rho}(x'+\rho\rho'x''+\rho^2x''') & N=\dfrac{\sigma}{\rho}(z'+\rho\rho'z''+\rho^2z''') \\ M=\dfrac{\sigma}{\rho}(y'+\rho\rho'y''+\rho^2y''') & P=\dfrac{\sigma}{\rho}(w'+\rho\rho'w''+\rho^2w''') \end{array}\right\} \quad (6)$$

The equations of the binormal plane can, therefore, be deduced from the fact that it passes through the

trinormal, *i.e.*, the normal to the osculating hyperplane, and the principal binormal. Thus, the equations of the binormal plane can also be written as

$$\begin{Vmatrix} X-x & Y-y & Z-z & W-w \\ L & M & N & P \\ T_x & T_y & T_z & T_w \end{Vmatrix}=0 \qquad (229.1$$

where T_x, T_y, T_z, T_w are the direction-consines of the trinormal (§ 226).

230. The Prinicpal Lines of a Curve:

There are then the following four lines associated with a curve at any point **P** :—

(1) The tangent,
(2) The principal normal,
(3) The principal binormal,
(4) The trinormal.

These four lines are perpendicular in pairs, forming an orthogonal frame at **P**. The tangent and the principal normal lie in the osculating plane; and the principal binormal and the trinormal lie in the binormal plane.

231. The Three Curvatures: *

The deviation or deflection of a curve from the tangent line, from the osculating plane and from the osculating hyperplane at any point may be called its " three curvatures."

* For analytical theory of three curvatures, see Pirondini, *Sulle line a triple curvature nello spazio Euclides a quatre dimensioni*—Giorni di Battaglini (1890). See also Forsyth, *loc. cit.*, Vol. I (1930), Chap. IX.

The deviation from the tangent line at any point is called the *prime curvature, or plane curvature*, and is measured by the ratio of the angle between the tangent lines at two consecutive points, *i.e.*, the *angle of contingence* to the infinitesimal arc between their points of contact. It is, in fact, the arc-rate of rotation of the tangent line. If d_ϵ denotes the angle of contingence and ds denotes the corresponding arc, then $\frac{d_\epsilon}{d_s}=\frac{1}{\rho}$ measures the prime or plane curvature of the curve at the point. ρ is called the *radius* of prime curvature, and is the radius of the osculating circle in the osculating plane, meeting the curve in three consecutive points.

The deflection of the curve from the osculating plane at any point is called the *second curvature*, or *torsion*, and is measured by the ratio of the angle between two consecutive osculating planes to the infinitesimal arc between their points of contact. In fact, *torsion* measures the arc-rate of turning of the osculating plane round the tangent line. If $d\eta$ denotes the angle between two consecutive osculating planes, *i.e.*, if $d\eta$ be the angle of torsion, and ds denotes the corresponding infinitesimal arc, then $\frac{d\eta}{ds}=\frac{1}{\sigma}$ measures the torsion, *i.e.*, the deflection of the curve from the osculating plane. By analogy to the radius of prime curvature, σ is called the *radius of torsion*, although it is simply a magnitude, having no direction, nor any organic relation with the curve.

The deflection of the curve from its osculating hyperplane at any point is called its *third curvature*, or *tilt*, and is measured by the ratio of the angle between

two consecutive osculating hyperplanes (*i.e.*, the angle between two consecutive trinormals) to the infinitesimal arc between their points of contact. If $d\omega$ denotes the angle between two consecutive osculating hyperplanes, *i.e.*, if $d\omega$ be the *angle of tilt* and ds the corresponding infinitesimal arc, then $\frac{d\omega}{ds}=\frac{1}{\tau}$ measures the *third curvature*, or *tilt*, and by analogy, τ may be called the *radius of tilt*. It is to be remarked that there is no centre of tilt, nor is there any line whose direction may be regarded as the radius of tilt.

232. The Prime or Circular Curvature :

We have seen that the normal hyperplane at P and the normal hyperplane at the consecutive point P′ intersect in a plane defined by the equations

$$\Sigma(X-x)x'=0, \quad \Sigma(X-x)x''=1. \quad \ldots \quad (1)$$

This plane is absolutely orthogonal to the osculating plane, but does not pass through the point P. It intersects the osculating plane at a point C (say).

Since C lies in the osculating plane, its co-ordinates may be taken as

$$X=x+\alpha x'+\beta x'', \quad Z=z+\alpha z'+\beta z'',$$

$$Y=y+\alpha y'+\beta y'', \quad W=w+\alpha w'+\beta w''.$$

If the point C lies in the plane (1), these values of the co-ordinates must satisfy its equations.

$$\therefore \quad \Sigma\{(x+\alpha x'+\beta x'')-x\}x'=0$$

$$\Sigma\{(x+\alpha x'+\beta x'')-x\}x''=1$$

$$\text{or,} \quad \alpha\Sigma x'^2+\beta\Sigma x'x''=0, \quad \alpha\Sigma x'x''+\beta\Sigma x''^2=1 \quad \ldots \quad (2)$$

Also, since $\Sigma x'^2=1$, we have $\Sigma x'x''=0$.

$\therefore$ These relations give

$$\alpha=0, \quad \text{and} \quad \Sigma x''^2=\frac{1}{\beta}=\frac{1}{\rho^2}, \qquad i.e., \quad \beta=\rho^2.$$

$\therefore$ From (2), the co-ordinates of C are

$$x+\rho^2 x'', \quad y+\rho^2 y'', \quad z+\rho^2 z'', \quad w+\rho^2 w''$$

and $$PC^2=\Sigma\{(x+\rho^2 x'')-x\}^2=\rho^4\Sigma x''^2=\rho^2.$$

Hence, we may take $PC=\rho$, and the direction-cosines of PC are given by $\rho x''$, $\rho y''$, $\rho z''$, $\rho w''$.

Since the direction-cosines of the tangent PT are x', y', z', w', and we have also

$$\rho x''.x' + \rho y''.y' + \rho z''.'z + \rho w''.w'=\rho\Sigma x'x''=0,$$

it follows that the line PC is at right angles to the tangent PT, and since it is in the osculating plane, PC is a *normal* to the curve.

Similarly, by considering the line joining C to the consecutive point P′, it can be shown that P′C is at right angles to the tangent at P′. Now, a circle can be drawn through P, P′ and P″, and in the limit, when P, P′, P″ approach to coincidence, chords PP′ and P′P″ become consecutive tangents, and consequently PC and P′C are two consecutive normals, intersecting at the centre C of the circle. Hence, this circle, passing through three consecutive points on the curve, is the osculating circle, and is called the *circle of curvature*, and C is the *centre of curvature*. The radius $PC=\rho$ is called the *radius of circular curvature*. Thus, the radius of circular curvature being denoted by ρ, the co-ordinates of the centre of circular curvature are

$$x+\rho^2 x'', \quad y+\rho^2 y'', \quad z+\rho^2 z'', \quad w+\rho^2 w''$$

where $$1/\rho^2=x''^2+y''^2+z''^2+w''^2. \qquad (232.1$$

Alternative Method :

Let $d\epsilon$ denote the angle between the tangent at P and the tangent at the consecutive point P′. The two tangents evidently lie in the osculating plane at P.

The direction-cosines of the tangent at P are x', y', z', w', and those of the tangent at P′ are

$$x'+x''ds,\ y'+y''ds,\ z'+z''ds,\ w'+w''ds.$$

$$\begin{aligned}\therefore\quad \sin^2 d\epsilon &= \Sigma\{x'(y'+y''ds)-y'(x'+x''ds)\}^2\\ &= \Sigma(x'y''-x''y')^2 ds^2\\ &= \{\Sigma x'^2\Sigma x''^2-(\Sigma x'x'')^2\}ds^2\\ &= \frac{1}{\rho^2}\,ds^2.\end{aligned}$$

Since $d\epsilon$ is a very small angle, $\sin d\epsilon = d\epsilon$, and in the limit, we have

$$\frac{1}{\rho^2}=\frac{d\epsilon^2}{ds^2},\ \ i.e.,\ \ \frac{1}{\rho}=\frac{d\epsilon}{ds}. \tag{232.2}$$

233. The Second Curvature or Torsion :

As already pointed out, a curve in general position in the fourfold can deviate from the osculating plane at any point, and this deviation is measured by the arc-rate of turning of the osculating plane about the tangent at the point. If, then, $d\eta$ denotes the angle between the osculating plane at any point P and that at a consecutive point P′, then the deviation is measured by $\dfrac{d\eta}{ds}$, and by analogy to the circular curvature, we may denote this ratio by $\dfrac{1}{\sigma}$, so that $\dfrac{d\eta}{ds}=\dfrac{1}{\sigma}$, where σ is called the radius of torsion,

or second curvature. It is to be noticed that unlike the centre of circular curvature, there is no point which may be called the centre of torsion, and there is no direction like that of the radius of curvature. σ is called, by analogy, *the radius of torsion*, but it is simply a magnitude and devoid of direction. The angle $d\eta$ may be called the *angle of torsion.*

Since the osculating planes at two consecutive points intersect in a line, $d\eta$ is the small dihedral angle between the two planes, and can, therefore, be measured, as in the ordinary space, by the angle between two lines drawn in the respective planes at right angles to the common line of section of the two planes. This common line is the tangent at the consecutive point P′.

The direction-cosines of the tangent at P′ are

$$x'+x''ds,\ y'+y''ds,\ z'+z''ds,\ w'+w''ds. \qquad \dots \quad (1)$$

Let l_1, m_1, n_1, p_1 be any direction in the osculating plane. Then

$$l_1=\alpha x'+\beta x'',\quad m_1=\alpha y'+\beta y'',$$

$$n_1=\alpha z'+\beta z'',\quad p_1=\alpha w'+\beta w''$$

where $\quad \Sigma l_1{}^2=1=\Sigma(\alpha x'+\beta x'')^2$

$$=\alpha^2\Sigma x'^2+\beta^2\Sigma x''^2=\alpha^2+\frac{\beta^2}{\rho^2}. \qquad \dots \quad (2)$$

If, then, this line is perpendicular to the common line (1), we must have

$$\Sigma l_1(x'+x''ds)=0, \quad \text{or,} \quad \Sigma(\alpha x'+\beta x'')(x'+x''ds)=0,$$

i.e., $\quad \alpha\Sigma x'^2+\beta\Sigma x''^2 ds=0, \qquad$ or, $\quad \alpha+\dfrac{\beta}{\rho^2}ds=0. \quad \dots \quad (3)$

From (2) and (3) we may take, since only the lowest power of ds is retained,

$$\alpha = -\frac{ds}{\rho}, \quad \text{and} \quad \beta = \rho. \qquad \dots \quad (4)$$

$\therefore$ The direction-cosines of the line in the osculating plane at right angles to the common line are

$$\rho x'' - x'\frac{ds}{\rho}, \quad \rho y'' - y'\frac{ds}{\rho}, \quad \rho z'' - z'\frac{ds}{\rho}, \quad \rho w'' - w'\frac{ds}{\rho}. \qquad \dots \quad (5)$$

Let l_2, m_2, n_2, p_2 be a direction in the osculating plane at the consecutive point P′, at right angles to the common line.

It is, then, the principal normal at P′, and its direction-cosines are, therefore,

$$l_2 = \rho x'' + (\rho x''' + \rho' x'')ds, \quad m_2 = \rho y'' + (\rho y''' + \rho' y'')ds$$
$$n_2 = \rho z'' + (\rho z''' + \rho' z'')ds, \quad p_2 = \rho w'' + (\rho w''' + \rho' w'')ds.$$

If ψ is the angle between the lines (l_1, m_1, n_1, p_1) and (l_2, m_2, n_2, p_2), we have

$$\begin{aligned}\sin^2 \psi &= \Sigma(l_1 m_2 - l_2 m_1)^2 \\ &= \Sigma\Big[\Big(\rho x'' - x'\frac{ds}{\rho}\Big)\{\rho y'' + (\rho y''' + \rho' y'')ds\} - \\ &\qquad \{\rho x'' + (\rho x''' + \rho' x'')ds\}\Big(\rho y'' - y'\frac{ds}{\rho}\Big)\Big]^2 \\ &= \Sigma[\{\rho^2(x''y''' - x'''y'') - (x'y'' - x''y')\}ds]^2\end{aligned}$$

retaining only the lowest power of ds.

$$\begin{aligned}\therefore \quad \frac{\sin^2 \psi}{ds^2} &= \Sigma\{x''(\rho^2 y''' + y') - y''(\rho^2 x''' + x')\}^2 \\ &= (\Sigma x''^2)\{\Sigma(\rho^2 x''' + x')^2\} - \{\Sigma x''(\rho^2 x''' + x')\}^2 \\ &= \frac{1}{\rho^2}\{\rho^4 \Sigma x'''^2 + 2\rho^2 \Sigma x' x''' + \Sigma x'^2\} \\ &\qquad - \{\rho^2 \Sigma x'' x''' + \Sigma x' x''\}^2\end{aligned}$$

$$=\rho^2\Sigma x'''^2+2\left(-\frac{1}{\rho^2}\right)+\frac{1}{\rho^2}-\left\{\rho^2.\left(-\frac{\rho'}{\rho^3}\right)\right\}^2$$

$$=\rho^2\Sigma x'''^2-\frac{1}{\rho^2}-\left\{-\frac{\rho'}{\rho}\right\}^2$$

$$=\frac{1}{\rho^2}\{\rho^4\Sigma x'''^2-1-\rho'^2\}=\frac{1}{\sigma^2}\quad\text{(say).}$$

$\therefore$ Since ψ is a very small angle, we have

$$\sin\psi=\psi=d\eta\qquad\therefore\qquad\frac{d\eta}{ds}=\frac{1}{\sigma},$$

where σ is called the *radius of torsion*. We have then

$$\rho^2\Sigma x'''^2-\frac{1}{\rho^2}-\frac{\rho'^2}{\rho^2}=\frac{1}{\sigma^2},\text{ or, }\rho^2\Sigma x'''^2=\frac{1}{\sigma^2}+\frac{1}{\rho^2}+\frac{\rho'^2}{\rho^2}$$

$$\therefore\quad\Sigma x'''=\frac{1}{\rho^2\sigma^2}+\frac{1}{\rho^4}+\frac{\rho'^2}{\rho^4}.\qquad(233.1$$

234. The Trinormal:

We are now in a position to express the direction-cosines of the trinormal in terms of the radii of circular curvature and torsion.

Writing the determinant equation (§ 226) in the form

$$L(X-x)+M(Y-y)+N(Z-z)+P(W-w)=0$$

the direction-cosines of the trinormal are proportional to L, M, N, P, and the actual direction-cosines may be taken as L, M, N, P, if they are subject to the relation

$$L^2+M^2+N^2+P^2=1.$$

The co-factors of $X-x$, etc. in the determinant are consequently proportional to L, M, N, P, and we may write

$$L=\Theta.T_x,\quad M=\Theta.T_y,\quad N=\Theta.T_z,\ P=\Theta.T_w,\ \text{where}$$

$$T_x=\begin{vmatrix} y' & z' & w' \\ y'' & z'' & w'' \\ y''' & z''' & w''' \end{vmatrix},\quad T_y=-\begin{vmatrix} x' & z' & w' \\ x'' & z'' & w'' \\ x''' & z''' & w''' \end{vmatrix},$$

$$T_z=\begin{vmatrix} x' & y' & w' \\ x'' & y'' & w'' \\ x''' & y''' & w''' \end{vmatrix},\quad T_w=-\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix},$$

and $L^2+M^2+N^2+P^2=\Theta^2.(T_x{}^2+T_y{}^2+T_z{}^2+T_w{}^2)=1.$

$$\therefore\ \frac{1}{\Theta^2}=\Sigma T_x{}^2=\begin{vmatrix} \Sigma x'^2 & \Sigma x'x'' & \Sigma x'x''' \\ \Sigma x'x'' & \Sigma x''^2 & \Sigma x''x''' \\ \Sigma x'x''' & \Sigma x''x''' & \Sigma x'''^2 \end{vmatrix}$$

$$=\begin{vmatrix} 1 & 0 & -\frac{1}{\rho^2} \\ 0 & \frac{1}{\rho^2} & -\frac{\rho'}{\rho^3} \\ -\frac{1}{\rho^2} & -\frac{\rho'}{\rho^3} & \frac{1}{\rho^2\sigma^2}+\frac{1}{\rho^4}+\frac{\rho'^2}{\rho^4} \end{vmatrix}$$

$$=\frac{1}{\sigma^2\rho^4}. \tag{234.1}$$

235. The third Curvature or Tilt:

As already stated, the deviation of the curve from the osculating hyperplane is measured by what is called its *tilt*. If the angle between the osculating

hyperplanes at two consecutive points be denoted by $d\omega$, then the arc-rate of variation is measured by

$$\frac{d\omega}{ds}=\frac{1}{\tau},$$

where τ is called the *radius of tilt*. It is to be noticed that τ is a linear magnitude, but there is no centre of tilt, and although the expression 'radius of tilt' is used, there is no direction which may be regarded as the radius of tilt.

The angle between osculating hyperplanes at two consecutive points is the same as that between their normals. If then l, m, n, p and l', m', n', p' are the direction-cosines of two consecutive trinormals, we have

$$l=\sigma\rho^2 T_x, \quad n=\sigma\rho^2 T_z, \quad l'=l+dl, \quad n'=n+dn,$$
$$m=\sigma\rho^2 T_y, \quad p=\sigma\rho^2 T_w, \quad m'=m+dm, \quad p'=p+dp.$$

If $d\omega$ denotes the angle between the two trinormals, we have

$$\begin{aligned}\sin^2 d\omega&=\Sigma(lm'-l'm)^2\\&=\Sigma\{l(m+dm)-(l+dl)m\}^2=(ldm-mdl)^2.\\&=\Sigma\{(\sigma\rho^2 T_x)\ d(\sigma\rho^2.T_y)-(\sigma\rho^2 T_y)\ d(\sigma\rho^2 T_x)\}^2\\&=\sigma^4\rho^8\Sigma(T_x dT_y-T_y dT_x)^2.\end{aligned}$$

$$\therefore\quad \frac{\sin^2 d\omega}{ds^2}=\sigma^4\rho^8\Sigma\left(T_x.\frac{dT_y}{ds}-T_y.\frac{dT_x}{ds}\right)^2.$$

Denoting by Ω the determinant

$$\Omega\equiv\begin{vmatrix} x' & y' & z' & w' \\ x'' & y'' & z'' & w'' \\ x''' & y''' & z''' & w''' \\ x^{iv} & y^{iv} & z^{iv} & w^{iv}\end{vmatrix}$$

Since $-T_x, T_y, \frac{dT_x}{ds}, \frac{dT_y}{ds}$ are the minors of

$$x^{iv}, \quad y^{iv}, \quad x^{iv}, \quad y^{iv},$$

we have
$$T_x \cdot \frac{dT_y}{ds} - T_y \cdot \frac{dT_x}{ds} = (z'w'' - z''w')\Omega.$$

$$\therefore \qquad \frac{\sin^2 d\omega}{ds^2} = \sigma^4\rho^8\Sigma(z'w'' - z''w')^2\Omega^2$$

$$\text{or,} \quad \left(\frac{d\omega}{ds}\right)^2 = \sigma^4\rho^8\Sigma(z'w'' - z''w')^2\Omega^2$$

$$= \sigma^4\rho^8\Omega^2\{\Sigma x'^2 . \Sigma x''^2 - (\Sigma x'x'')^2\}$$

$$= \sigma^4\rho^8\Omega^2\left\{\frac{1}{\rho^2}\right\} = \sigma^4\rho^6\Omega^2$$

$$\therefore \quad \frac{d\omega}{ds} = \frac{1}{\tau} = \sigma^2\rho^3\Omega, \tag{235.1}$$

taking the positive sign of the square root.

Taking the positive square root (§ 234), we have $\Theta = \sigma\rho^2$, and the direction-cosines of the trinormal are

$$\sigma\rho^2 T_x, \quad \sigma\rho^2 T_y, \quad \sigma\rho^2 T_z, \quad \sigma\rho^2 T_w.$$

This line lies in the binormal plane and is perpendicular to the tangent line, the principal normal and the binormal, and is accordingly called the *trinormal*.

236. Significance of Curvature, Torsion and Tilt.

It is now clear that a skew curve has three varieties of curvatures—Prime curvature $1/\rho$, Torsion $1/\sigma$ and Tilt $1/\tau$,—and these are not generally infinite.

When ρ is infinite everywhere along a curve, there is no plane curvature anywhere, and the curve is a *straight line*.

When σ is infinite everywhere along the curve, there is no torsion anywhere, *i.e.*, the curve does not deflect from the osculating plane, and it is consequently a *plane curve*.

When τ is infinite everywhere, there is no tilt anywhere, the osculating hyperplane is everywhere the same, *i.e.*, the curve always lies in one and the same hyperplane, and it is a *space-curve*.

Again, a plane curve of constant curvature is a circle, so that when ρ is constant, the curve is a *circle*. If a space-curve has the ratio of its curvature and torsion constant, it is a cylindrical helix, and when both are constant, the curve is a *circular helix*.* A curve in the fourfold may have all its three curvatures constant. Such a curve is a geodesic on a *cylindro-cylindric* hypersurface, and is uniquely determinate except as to position and orientation.† If, however, the ratios of these curvatures are constant, the curve may be taken as a geodesic in a sphero-cylindrical hypersurface.‡

237 The principal Lines at a consecutive Point P′, referred to the principal Lines at P.

Let the direction-cosines of the four principal lines at any point P be denoted by

$l_1,\ m_1,\ n_1,\ p_1$—the tangent,

$l_2,\ m_2,\ n_2,\ p_2$—the principal normal.

$l_3,\ m_3,\ n_3,\ p_3$—the principal binormal.

$l_4,\ m_4,\ n_4,\ p_4$—the trinormal.

* Salmon, *Analytical Geometry of Three Dimensions* (1912) p. 389.

† Forsyth, *loc. cit.*, Vol. 1 (1930), p. 284.

‡ Forsyth, *loc. cit.*, Vol. 2 (1930), p. 87.

The direction-cosines of the principal lines at any consecutive point P′, referred to the principal lines at P, are denoted by the corresponding accented letters. Then we have

$l'_1,\ m'_1,\ n'_1,\ p'_1$ (tangent at P′) $=1,\ d\epsilon,\ 0,\ 0.$

For, the tangent at P′ makes an infinitesimal angle $d\epsilon$ with the tangent at P, and also lies in the osculating plane at P. Hence, it makes an angle $d\epsilon,\ \frac{1}{2}\pi-d\epsilon,\ \frac{1}{2}\pi,$ $\frac{1}{2}\pi$ with the tangent, the principal normal, the principal binormal and trinormal at P, and consequently, its direction-cosines referred to those lines are

$$\cos d\epsilon,\ \cos(\tfrac{1}{2}\pi-d\epsilon).,\ 0,\ 0,\quad i.e.,\quad 1,\ d\epsilon,\ 0,\ 0. \qquad \ldots \quad (1)$$

The principal normal at P′ lies in the osculating plane at P, and is perpendicular to the tangent at P′. Hence, it makes an angle $(\frac{1}{2}\pi+d\epsilon)$ with the tangent at P, and also an infinitesimal angle $d\epsilon$ with the principal normal at P. Again, it lies in the osculating hyperplane at P and also in the osculating plane at P. Hence, it makes an angle $(\frac{1}{2}\pi-d\eta)$ with the binormal at P. It is evidently perpendicular to the trinormal. Hence, the direction-cosines of the principal normal at P′, referred to the principal lines at P, are

$l'_2=\cos(\frac{1}{2}\pi+d\epsilon)=-\sin d\epsilon=-d\epsilon$ (retaining small magnitudes of the first order)

$m'_2=\cos d\epsilon=1,$

$$n'_2=\cos(\tfrac{1}{2}\pi-d\eta)=\sin d\eta=d\eta, \qquad \ldots \quad (2)$$

$$p'_2=\cos\tfrac{1}{2}\pi=0.$$

The principal binormal $(l'_3,\ m'_3,\ n'_3,\ p'_3)$ at P′ lies in the binormal plane and the osculating hyperplane. It makes an infinitesimal angle $d\eta$ with the principal binormal at P, and an angle $(\frac{1}{2}\pi+d\eta)$ with the principal normal at P, while it is certainly at right angles to the

tangent at P. Since it lies in the osculating hyperplane at P′, it makes an angle $\frac{1}{2}\pi - d\omega$ with the trinormal at P. Hence

$$l'_3 = 0, \quad m'_3 = \cos(\tfrac{1}{2}\pi + d\eta) = -\sin d\eta = -d\eta,$$

$$n'_3 = \cos d\eta = 1, \quad p'_3 = \cos(\tfrac{1}{2}\pi - d\omega) = \sin d\omega = d\omega. \quad \ldots \quad (3)$$

The trinormal (l'_4, m'_4, n'_4, p'_4) at P′ is perpendicular to the tangent and the principal normal at P, and makes an angle $(\frac{1}{2}\pi + d\omega)$ with the principal binormal and an angle $d\omega$ with the trinormal at P. Hence

$$l'_4 = 0, \quad n'_4 = \cos(\tfrac{1}{2}\pi + d\omega) = -\sin d\omega = -d\omega,$$

$$m'_4 = 0, \quad p_4' = \cos d\omega = 1. \quad \ldots \quad (4)$$

238. Extension of Frenet-Serret Formulæ :*

Having determined the principal directions associated with a curve at two consecutive points and the three curvatures, it is now found possible to extend Frenet-Serret Formulæ to skew curves in the fourfold.

Let $l_i + dl_i, \quad m_i + dm_i, \quad n_i + dn_i, \quad p_i + dp_i,$

$$(i = 1, 2, 3, 4)$$

represent the direction-cosines of the four principal lines at a consecutive point P′. From what has been found above, we obtain

$$l_i + dl_i = l_i . 1 + m_i . d\epsilon + n_i . 0 + p_i . 0$$

$$= l_i + m_i . d\epsilon, \quad \text{or,} \quad dl_i = m_i . d\epsilon$$

$$\text{i.e.,} \quad \frac{dl_i}{ds} = m_i . \frac{d\epsilon}{ds} = \frac{m_i}{\rho} . \quad (238.1$$

* Frenet, *Liouville's Journal* t. XVII (1852), p. 437, and Serret, *ibid*, t. XVI (1851), p. 193.

$$m_i + dm_i = l_i.(-d\epsilon) + m_i.1 + n_i.d\eta + p_i.0$$

$$= -l_i.d\epsilon + m_i + n_i.d\eta$$

or, $$dm_i = -l_i.d\epsilon + n_i.d\eta$$

i.e., $$\frac{dm_i}{ds} = -l_i.\frac{d\epsilon}{ds} + \frac{d\eta}{ds} = -\frac{l_i}{\rho} + \frac{n_i}{\sigma}. \qquad (238.2$$

Again, $$n_i + dn_i = l_i.0 + m_i.(-d\eta) + n_i.1 + p_i.d\omega$$

$$= -m_i.d\eta + n_i + p_i.d\omega$$

or, $$dn_i = -m_i d\eta + p_i.d\omega$$

i.e., $$\frac{dn_i}{ds} = -m_i.\frac{d\eta}{ds} + p_i.\frac{d\omega}{ds} = -\frac{m_i}{\sigma} + \frac{p_i}{\tau}. \qquad (238.3$$

Similarly, $$p_i + dp_i = l_i.0 + m_i.0 + n_i.(-d\omega) + p_i.1$$

$$= -n_i.d\omega + p_i, \quad \text{or,} \quad dp_i = -n_i d\omega,$$

whence $$\frac{dp_i}{ds} = -\frac{n_i}{\tau}. \qquad (238.4$$

These formulæ can also be deduced from the values of the direction-cosines of the four principal lines already obtained. These, in fact, are obtained from the formulæ of transformation of co-ordinates as applied in the fourfold.

239. Expansion of the Co-ordinates:

Taking the origin on the curve and the tangent, the principal normal, the principal binormal and the tri-normal as the axes of reference, the co-ordinates of a point P near the origin, at an arcual distance s, can be

expanded in powers of s by Maclaurin's theorem as follows :

If x, y, z, w, are the co-ordinatesof P, we may write

$$x=f(s)=f(o)+sf'(o)+\frac{s^2}{2!}f''(o)+\frac{s^3}{3!}f'''(o)+\frac{s^4}{4!}f^{iv}(o)+\ldots\ldots\ldots$$

$$=sx_0'+\frac{s^2}{2!}x_0''+\frac{s^3}{3!}x_0'''+\frac{s^4}{4!}x_0^{iv}+\ldots\ldots\ldots$$

where $x_0', x_0'', x_0''', x_0^{iv}$......are the values of x', x'', x''', x^{iv},..........at the origin.

Similarly, $y=sy_0'+\frac{s^2}{2!}y_0''+\frac{s^3}{3!}y_0'''+\frac{s^4}{4!}y^{iv}+\ldots\ldots$

$$z=sz_0'+\frac{s^2}{2!}z_0''+\frac{s^3}{3!}z_0'''+\frac{s^4}{4!}z^{iv}+\ldots\ldots$$

$$w=sw_0'+\frac{s^2}{2!}w_0''+\frac{s^3}{3!}w_0'''+\frac{s^4}{4!}w^{iv}+\ldots\ldots$$

Since the tangent is the axis of x, we have

$$x_0'=1, \quad y_0'=0, \quad z_0'=0, \quad w_0'=0. \qquad \ldots \quad (1)$$

Since the principal normal is the y-axis, we have

$$\rho x_0''=0, \quad \rho y_0''=1, \quad \rho z_0''=0, \quad \rho w_0''=0. \ \ldots \quad (2)$$

Since the principal binormal is the axis of z, we have

$\frac{\sigma}{\rho}(x'+\rho\rho' x''+\rho^2 x''')=0$, whence $\frac{\sigma}{\rho}(x_0'+\rho\rho' x_0''+\rho^2 x_0''')=0$,

i.e., $\rho^2 x_0'''=-x_0'-\rho\rho' x_0''=-1. \quad \therefore\ x_0'''=-\frac{1}{\rho^2}$.

Similarly, $y_0'+\rho\rho' y_0''+\rho^2 y_0'''=0$, whence

$$\rho^2 y_0'''=-(y_0'+\rho\rho' y_0'')=-\rho'. \quad \therefore\ y_0'''=-\frac{\rho'}{\rho^2}.$$

Also,

$\frac{\sigma}{\rho}(z'_0+\rho\rho' z''_0+\rho^2 z_0''')=1$, or, $z_0'+\rho\rho' z_0''+\rho^2 z_0'''=\frac{\rho}{\sigma}$.

$$\therefore \quad \rho^2 z_0''' = \frac{\rho}{\sigma} - z_0' - \rho\rho' z_0'' = \frac{\rho}{\sigma}. \quad \therefore \quad z_0''' = \frac{\rho}{\sigma\rho^2} = \frac{1}{\sigma\rho}.$$

Similarly, $w_0' + \rho\rho' w_0'' + \rho^2 w_0''' = 0$, whence

$$\rho^2 w_0''' = -w_0' - \rho\rho' w_0'' = 0, \quad i.e., \quad w_0''' = 0.$$

$$\therefore \quad x_0''' = -\frac{1}{\rho^2}, \quad y_0''' = -\frac{\rho'}{\rho^2}, \quad z_0''' = \frac{1}{\sigma\rho}, \quad w_0''' = 0 \; \ldots \quad (3)$$

The values of x_0^{iv}, y_0^{iv}, z_0^{iv}, w_0^{iv} can be calculated from the extended forms of Frenet's Formulæ. Using the third equation, we have

$$\frac{p_i}{\tau} = \frac{m_i}{\sigma} + \frac{dn_i}{ds} \qquad \ldots \quad (4)$$

and since the trinormal is the axis of w, we have

$$p_1 = 0, \; p_2 = 0, \; p_3 = 0, \; p_4 = 1,$$

whence, $\frac{\rho}{\sigma} x'' + \frac{d}{ds}\left\{\frac{\sigma}{\rho}(x' + \rho\rho' x'' + \rho^2 x''')\right\} = 0,$

$$\frac{\rho}{\sigma} y'' + \frac{d}{ds}\left\{\frac{\sigma}{\rho}(y' + \rho\rho' y'' + \rho^2 y''')\right\} = 0,$$

$$\frac{\rho}{\sigma} z'' + \frac{d}{ds}\left\{\frac{\sigma}{\rho}(z' + \rho\rho' z'' + \rho^2 z''')\right\} = 0,$$

$$\frac{\rho}{\sigma} w'' + \frac{d}{ds}\left\{\frac{\sigma}{\rho}(w' + \rho\rho' w'' + \rho^2 w''')\right\} = \frac{1}{\tau},$$

whence, by substituting the values of x'_0, y_0', $\ldots x_0'' \ldots$ the values of x_0^{iv}, y_0^{iv}, z_0^{iv} and w_0^{iv} can be calculated.

We find

$$x_0{}^{iv} = \frac{3\rho'}{\rho^3},\ y_0{}^{iv} = 2\frac{\rho'^2}{\rho^3} - \frac{\rho''}{\rho^2} - \frac{1}{\rho^3} - \frac{1}{\rho\sigma^2},$$

$$z_0{}^{iv} = -\frac{1}{\rho\sigma}\left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right),\ w_0{}^{iv} = \frac{1}{\rho\sigma\tau}.$$

A general formula for determining the fourth order derivatives of x, y, z and w can be calculated easily from (4). We have

$$p_1 = \sigma\rho^2.\mathrm{T}_x,\quad p_2 = \sigma\rho^2.\mathrm{T}_y,\quad p_3 = \sigma\rho^2.\mathrm{T}_z,\quad p_4 = \sigma\rho^2.\mathrm{T}_w$$

$$\therefore\quad \frac{\sigma\rho^2\mathrm{T}_x}{\tau} = \frac{\rho}{\sigma}x'' + \frac{d}{ds}\left\{\frac{\sigma}{\rho}x' + \rho\rho'x'' + \rho^2 x'''\right\}$$

$$= \frac{\sigma}{\rho'}\left(\frac{\sigma'}{\sigma} - \frac{\rho'}{\rho}\right)x' + \left\{\frac{\rho}{\sigma} + \frac{\sigma}{\rho} + \frac{d}{ds}(\sigma\rho')\right\}x''$$

$$+ (2\rho'\sigma + \rho\sigma')x''' + \rho\sigma x^{iv}$$

which finally gives

$$x^{iv} = \frac{\rho}{\tau}\mathrm{T}_x - \frac{1}{\rho^2}\left(\frac{\sigma'}{\sigma} - \frac{\rho'}{\rho}\right)x' - \left\{\frac{1}{\rho^2} + \frac{1}{\sigma^2} + \frac{1}{\rho\sigma}\frac{d}{ds}(\sigma\rho')\right\}x''$$

$$-\left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right)x'''.$$

Similar expressions are obtained for y^{iv}, z^{iv} and w^{iv}. Hence, finally, the co-ordinates of any point P in the neighbourhood of the origin, at an arcual distance s, are expressed in the forms

$$x = s - \frac{1}{6\rho^2}s^3 + \frac{1}{8}\frac{\rho'}{\rho^3}s^4 + \ldots\ldots$$

$$y = \frac{1}{2\rho}s^2 - \frac{1}{6}.\frac{\rho'}{\rho^2}s^3 + \frac{1}{24}\left(2\frac{\rho'^2}{\rho^3} - \frac{\rho''}{\rho^2} - \frac{1}{\rho^3} - \frac{1}{\rho\sigma^2}\right)s^4 + \ldots$$

$$z = \frac{1}{6\rho\sigma}s^3 - \frac{1}{24\rho\sigma}\left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right)s^4 + \ldots\ldots$$

$$w = \frac{1}{24}.\frac{s^4}{\rho\sigma\tau} + \text{higher powers of } s. \tag{239.1}$$

Using these expansions of co-ordinates, we can now determine the projection of the arc PQ on the principal lines at any point P.

The projection of PQ on the tangent line at P= sum of the projections on the tangent line of its projections on the co-ordinate axes

$$=\Sigma(X-x)x'=s-\frac{1}{6\rho^2}s^3+\frac{1}{8}\frac{\rho'}{\rho^3}s^4+\ldots \quad \ldots \quad (5)$$

Similarly, the projection on the principal normal

$$=\Sigma(X-x)\,\rho x''=\frac{1}{2\rho}s^2-\frac{1}{6}\cdot\frac{\rho'}{\rho^2}s^3$$

$$+\frac{1}{24}\left(\frac{2\rho'^2}{\rho^3}-\frac{\rho''}{\rho^2}-\frac{1}{\rho^3}-\frac{1}{\rho\sigma^2}\right)s^4+\ldots \quad \ldots \quad (6)$$

The projection of PQ on the principal binormal

$$=\Sigma(X-x).\frac{\sigma}{\rho}(x'+\rho\rho'x''+\rho^2x''')$$

$$=\frac{1}{6\rho\sigma}s^3-\frac{1}{24\rho\sigma}\left(2\frac{\rho'}{\rho}+\frac{\sigma'}{\sigma}\right)s^4\cdot\ldots \quad \ldots \quad (7)$$

The projection of PQ on the trinormal

$$=\Sigma(X-x).\,\sigma\rho^2T_x$$

$$=\sigma\rho^2\Sigma(X-x)\begin{vmatrix} y' & z' & w' \\ y'' & z'' & w'' \\ y''' & z''' & w''' \end{vmatrix}$$

$$=\frac{1}{24}\sigma\rho^2s^4\,\Sigma x^{iv}\begin{vmatrix} y' & z' & w' \\ y'' & z'' & w'' \\ y''' & z''' & w''' \end{vmatrix}+\text{higher powers of } s.$$

$$= -\frac{1}{24} \cdot \sigma\rho^2 . s^4 \, \Omega + \text{higher powers of } s \quad (\S\ 235)$$

$$= \frac{1}{24} \cdot \frac{s^4}{\rho\sigma\tau} + \text{higher powers of } s. \qquad \ldots \quad (8)$$

Hence, it is clear that if P is taken at the origin, and the principal lines are taken as the co-ordinate axes, the expansions already found easily follow from these expressions.

240. Spherical Curvature or Curvature of Torsion :

The sphere of closest contact at any point of a curve, *i.e.*, the sphere, determined by four consecutive points of a curve, is called the *osculating sphere* at the point, and may be used for determining the *spherical curvature* of a curve. The radius and centre of the osculating sphere are respectively called the *radius* and *centre* of spherical curvature of the curve.

If P, Q, R, S are four consecutive points of a curve, the limiting position of the sphere PQRS, as Q, R and S tend to P, is the *osculating sphere* at P. This sphere evidently lies in the osculating hyperplane at P.

The normal hyperplane at P intersects the normal hyperplane at the consecutive point P′ in a plane through the centre of circular curvature. The normal hyperplane at a further consecutive point P″ intersects that at P′ in a plane, which intersects the former plane in a line, *i.e.*, the normal hyperplanes at three consecutive points intersect in a line. This line intersects the osculating hyperplane through

three consecutive tangents in a point, which is a centre of curvature in the osculating hyperplane, and may be called the centre of spherical curvature.

The normal hyperplanes at P and P′ are respectively given by

$$\Sigma(X-x)x'=0, \qquad \Sigma(X-x)x''=1.$$

The normal hyperplanes at P′ and P″ (arc P′P″ being denoted by $d\sigma$) are

$$\Sigma(X-x)x'+\{\Sigma(X-x)x''-\Sigma x'^2\}d\sigma=0$$

$$\Sigma(X-x)x''+\{\Sigma(X-x)x''-\Sigma x'x''\}d\sigma=0$$

i.e., the intersection of the three normal hyperplanes is given by

$$\Sigma(X-x)x'=0, \quad \Sigma(X-x)x''=1, \quad \Sigma(X-x)x'''=0. \quad \ldots \quad (1)$$

The osculating hyperplane at P is

$$\begin{vmatrix} X-x & Y-y & Z-z & W-w \\ x' & y' & z' & w' \\ x'' & y'' & z'' & w'' \\ x''' & y''' & z''' & w''' \end{vmatrix}=0. \quad \ldots \quad (2)$$

Any point in this hyperplane is given by

$$X-x=\lambda x'+\mu x''+\nu x''', \text{ etc.}$$

Hence, these must satisfy the equations (1), and

we get $\lambda\Sigma x'^2+\mu\Sigma x'x''+\nu\Sigma x'x'''=0$

$$\text{or,} \qquad \lambda-\nu\frac{1}{\rho^2}=0. \qquad \ldots \quad (3)$$

Again, $\lambda\Sigma x'x'' + \mu x''^2 + \nu\Sigma x''x''' = 1$,

whence, $$\mu \frac{1}{\rho^2} - \nu . \frac{\rho'}{\rho^3} = 1. \quad \dots \quad (4)$$

Also, $\lambda\Sigma x'x''' + \mu\Sigma x''x''' + \nu\Sigma x'''^2 = 0$,

or, $$-\lambda \frac{1}{\rho^2} - \mu . \frac{\rho'}{\rho^3} + \nu \left(\frac{1}{\sigma^2\rho^2} + \frac{1}{\rho^4} + \frac{\rho'^2}{\rho^4}\right) = 0. \quad \dots \quad (5)$$

From (3), (4) and (5), we easily obtain

$$\lambda = \sigma^2 . \frac{\rho'}{\rho}, \quad \mu = \rho^2 + \sigma^2\rho'^2, \quad \nu = \sigma^2\rho\rho'.$$

$\therefore$ The co-ordinates of the centre of spherical curvature are

$$X = x + \sigma^2\frac{\rho'}{\rho}x' + (\rho^2 + \sigma^2\rho'^2)x'' + \sigma^2\rho\rho'x''', \text{ etc.} \quad (240.1$$

The radius R of spherical curvature is obtained by

$$R^2 = \Sigma(X - x)^2 = \Sigma \left\{\sigma^2 \frac{\rho'}{\rho}x' + (\rho^2 + \sigma^2\rho'^2)x'' + \sigma^2\rho\rho'x'''\right\}^2$$

$$= \rho^2 + \sigma^2\rho'^2. \quad (240.2$$

These results immediately lead to the following geometrical properties :—

Let C $(\alpha, \beta, \gamma, \delta)$ be the centre of circular curvature, and $S(\alpha', \beta', \gamma', \delta')$ the centre of spherical curvature at any point $P(x, y, z, w)$ of a curve.

Then, $\alpha = x + \rho^2 x''$, $\beta = y + \rho^2 y''$,

$\gamma = z + \rho^2 z''$, $\delta = w + \rho^2 w''$.

$\therefore \alpha' - \alpha = \sigma^2 \frac{\rho'}{\rho}(x' + \rho\rho'x'' + \rho^2 x''')$, with similar expressions

for $\beta'-\beta$, $\gamma'-\gamma$ and $\delta'-\delta$.

i.e., the quantities $\alpha'-\alpha$, $\beta'-\beta$,... are proportional to the direction-cosines of the principal binormal at P.

$$\therefore \quad \Sigma(\alpha'-\alpha)^2 = \Sigma\sigma^4 \frac{\rho'^2}{\rho^2}(x'+\rho\rho' x''+\rho^2 x''')^2$$

$$= \sigma^2\rho'^2 \Sigma \left\{ \frac{\sigma}{\rho}(x'+\rho\rho' x''+\rho^2 x''') \right\}^2$$

$$= \sigma^2\rho'^2, \qquad (240.3$$

since $\dfrac{\sigma}{\rho}(x'+\rho\rho' x''+\rho^2 x''')$, etc. are the actual direction-cosines of the binormal.

Also, C, the centre of circular curvature is the projection of the centre of spherical curvature on the osculating plane.

Hence, $\angle$ PCS = a right angle, and PC $=\rho$, CS $=\sigma\rho'$.

$$\therefore \quad PS^2 = R^2 = \rho^2 + \sigma^2\rho'^2 = PC^2 + CS^2.$$

The equation of the osculating sphere can also be obtained as the intersection of the osculating hyperplane at P and the hypersphere

$$\Sigma(X-x)^2 = 2\Sigma\{(\alpha'-x)(X-x)\}. \qquad (240.4$$

For, we may write

$$\Sigma(x-\alpha')^2 = \Sigma\{(X-x)-(\alpha'-x)\}^2$$

$$= \Sigma(X-x)^2 - 2\Sigma(X-x)(\alpha'-x) + \Sigma(\alpha'-x)^2$$

$$= PS^2 = \Sigma(\alpha'-x)^2.$$

241. Hyperspherical Curvature or Curvature of Tilt:

Any five consecutive points on a curve will determine a hypersphere, the centre and radius of which may be called the *centre* and the *radius* of hyperspherical curvature at the point.

Assume $(X-\alpha'')^2+(Y-\beta'')^2+(Z-\gamma'')^2+(W-\delta'')^2=\Gamma^2$ (1)

as the equation of the hypersphere. This is to pass through five consecutive points on the curve.

Differentiating (1) four times in succession, we get

$$\Sigma(X-\alpha'')x'=0, \qquad \Sigma(X-\alpha'')x''=-\Sigma x'^2=-1,$$

$$\Sigma(X-\alpha'')x'''=0, \qquad \Sigma(X-\alpha'')x^{iv}=-\Sigma x'x'''=\frac{1}{\rho^2}.$$

These five relations are sufficient to determine the five unknown quantities Γ, α'', β'', γ'', δ''.

Solving the last four, $X-\alpha''$, etc. can be determined, and then from the first, Γ is obtained.

Combining these equations with those giving the centre of spherical curvature, we find at once

$$(\alpha''-\alpha')x'+(\beta''-\beta')y'+(\gamma''-\gamma')z'+(\delta''-\delta')w'=0, \text{ etc.}$$

i.e., $\Sigma(\alpha''-\alpha')x'=0$, $\Sigma(\alpha''-\alpha')x''=0$, $\Sigma(\alpha''-\alpha')x'''=0$

whence, $$\frac{\alpha''-\alpha'}{T_x}=\frac{\beta''-\beta'}{T_y}=\frac{\gamma''-\gamma'}{T_z}=\frac{\delta''-\delta'}{T_w}.$$

But the normal to the osculating hyperplane through the centre of spherical curvature S is given by

$$\frac{X-\alpha'}{T_x}=\frac{Y-\beta'}{T_y}=\frac{Z-\gamma'}{T_z}=\frac{W-\delta'}{T_w}$$

which shows that $(\alpha'', \beta'', \gamma'', \delta'')$ is a point on this line, *i.e.*, the centre of hyperspherical curvature H lies on the normal to the osculating hyperplane through the centre of spherical curvature.

Ex. 1. Prove that PC is the principal normal to the curve at P.

Ex. 2. Show that if Ω vanishes at every point of the curve, it is a skew curve in a hyperplane.

Ex. 3. Show that a binormal is the line of intersection of the normal plane at P and the normal plane at a consecutive point.

Ex. 4. Prove that the line joining the centre of spherical curvature with that of the hyperspherical curvature is parallel to the trinormal, and every point on this line is equidistant from the four consecutive points P, P′, P″, P‴.

Ex. 5. Prove that the centre of hyperspherical curvature is the point of intersection of normal hyperplanes at four consecutive points.

Ex. 6. Prove that

$$\Gamma^2 = R^2 + \left(\frac{\tau}{\sigma} R \frac{dR}{d\rho} \right)^2.$$

Ex. 7. Show that the section of the osculating hypersphere by the osculating hyperplane is the osculating sphere, and that by the osculating plane is the circle of curvature.

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